

# Risk-sensitive Nonzero-sum Stochastic Differential Game with Unbounded Coefficients

Said Hamadène<sup>\*</sup> and Rui Mu<sup>†‡</sup>

December 4, 2014

## Abstract

This article is related to risk-sensitive nonzero-sum stochastic differential games in the Markovian framework. This game takes into account the attitudes of the players toward risk and the utility is of exponential form. We show the existence of a Nash equilibrium point for the game when the drift is no longer bounded and only satisfies a linear growth condition. The main tool is the notion of backward stochastic differential equation, which in our case, is multidimensional with continuous generator involving both a quadratic term of  $Z$  and a stochastic linear growth component with respect to  $Z$ .

**Keywords:** risk-sensitive; nonzero-sum stochastic differential games; Nash equilibrium point; backward stochastic differential equations.

**AMS subject classification:** 49N70; 49N90; 91A15.

## 1 Introduction

We consider, in this article, a risk-sensitive nonzero-sum stochastic differential game model. Assume that we have a system which is controlled by two players. Each one imposes an admissible control which is an adapted stochastic process denoted by  $u = (u_t)_{t \leq T}$  (resp.  $v = (v_t)_{t \leq T}$ ) for player 1 (resp. player 2). The state of the system is described by a process  $(x_t)_{t \leq T}$  which is the solution of the following stochastic differential equation:

$$dx_t = f(t, x_t, u_t, v_t)dt + \sigma(t, x_t)dB_t \text{ for } t \leq T \text{ and } x_0 = x, \quad (1.1)$$

where  $B$  is a Brownian motion. We establish this game model in a two-player framework for an intuitive comprehension. All results in this article are applicable to the multiple players case. Naturally, the control action is not free and has some risks. A *risk-sensitive nonzero-sum stochastic differential game* is a game model which takes into account the

---

<sup>\*</sup>Université du Maine, LMM, Avenue Olivier Messiaen, 72085 Le Mans, Cedex 9, France. hamadene@univ-lemans.fr.

<sup>†</sup>Université du Maine, LMM, Avenue Olivier Messiaen, 72085 Le Mans, Cedex 9, France; School of Mathematics, Shandong University, Jinan 250100, China. rui.mu.sdu@gmail.com.

<sup>‡</sup>Supported in part by the Natural Science Foundation for Young Scientists of Jiangsu Province, P.R. China (No. BK20140299).

attitudes of the players toward risk. More precisely speaking, for player  $i = 1, 2$ , the utility (cost or payoff) is given by the following exponential form

$$J^i(u, v) = \mathbf{E}[e^{\theta \{\int_0^T h_i(s, x_s, u_s, v_s) ds + g^i(x_T)\}}].$$

The parameter  $\theta$  represents the attitude of the player with respect to risk. What we are concerned here is a nonzero-sum stochastic differential game which means that the two players are of cooperate relationship. Both of them would like to minimize the cost and no one can cut more by unilaterally changing his own control. Therefore, the objective of the game problem is to find a *Nash equilibrium point*  $(u^*, v^*)$  such that,

$$J^1(u^*, v^*) \leq J^2(u, v^*) \text{ and } J^2(u^*, v^*) \leq J^2(u^*, v),$$

for any admissible control  $(u, v)$ .

Let us illustrate now, why  $\theta$ , in the cost function, can reflect the risk attitude of the controller. From the economic point of view, we denote by  $G_{u,v}^i = \int_0^T h_i(s, x_s, u_s, v_s) ds + g^i(x_T)$  the wealth of each controller and for a smooth function  $F(z)$ , let  $F(G_{u,v}^i)$  be the cost might be brought from the wealth. The two participates would like to minimize the expected cost  $\mathbf{E}[F(G_{u,v}^i)]$ . A notion of *risk sensitivity* is proposed as follows:

$$\gamma = \frac{F''(G^i)}{F'(G^i)}.$$

It is a reasonable function to reflect the trend, more precise, the curvature of cost  $F$  with respect to the wealth  $G^i$ . See W.H. Fleming's work [7] for more details. In the present paper, we choose the utility function  $F(z)$  as an exponential form  $F(z) = e^{\theta z}$ . Both theoretical and practical experiences have shown that it is often appropriate to use an exponential form of utility function. Therefore, the risk sensitivity  $\gamma$  is exactly the parameter  $\theta$ .

We explain this specific case  $\gamma = \theta$  in the following way. The expected utility  $J^i(u, v) = \mathbf{E}[e^{\theta G_{u,v}^i}]$  is certainty equivalent to

$$\varrho_\theta^i(u, v) := \theta^{-1} \ln \mathbf{E}[e^{\theta G_{u,v}^i}].$$

By certainty equivalent, we refer to the minimum premium we are willing to pay to insure us against some risk (or the maximum amount of money we are willing to pay for some gamble). Then,  $\varrho_\theta^i(u, v) \sim \mathbf{E}[G_{u,v}^i] + \frac{\theta}{2} \text{Var}(G_{u,v}^i)$  provided that  $\theta \text{Var}(G_{u,v}^i)$  is small ( $\text{Var}(\cdot)$  is the variance). Hence, minimizing  $J^i(u, v)$  is equivalent to minimize  $\varrho_\theta^i(u, v)$ . The variance  $\text{Var}(G_{u,v}^i)$  of the wealth reflects the risk of decision to a certain extent. Therefore, it is obvious that when  $\theta > 0$ , the less risk the better. Such a decision maker in economic markets will have a *risk-averse* attitude. On the contrary, when  $\theta < 0$ , the optimizer is called *risk-seeking*. Finally, if  $\theta = 0$ , this situation corresponds to the risk-neutral controller. Without loss of generality, we set  $\theta = 1$  in this work.

About the risk-sensitive stochastic differential game problem, including nonzero-sum, zero-sum and mean-field cases, there are some previous works. Readers are referred to [3, 5, 8, 9, 15, 23] for further acquaintance. Among those results, a particular popular approach is partial differential equation, such as [3, 8, 9, 15, 23] with various objectives.

Another method is through backward stochastic differential equation (BSDE) theory, see [5]. The nonlinear BSDE is introduced by Pardoux and Peng [18] and developed rapidly in the past two decades. The notion of BSDE is proved as an efficient tool to deal with stochastic differential game. It has been used in the risk neutral case, see [13, 12]. About Other applications such as in the field of mathematic finance, we refer the work by El-Kaoui et al. [6] (1997). A complete review on BSDEs theory as well as some new results on nonlinear expectation are introduced in a survey paper by Peng (2010) [20].

In the present paper, we study the risk-sensitive nonzero-sum stochastic differential game problem through BSDE in the same line as article by El-Karoui and Hamadène [5]. However in [5], the setting of game problem concerns only the case when the drift coefficient  $f$  in diffusion (1.1) is bounded. This constrain is too strict to some extent. Therefore, our motivation is to relax as much as possible the boundedness of the coefficient  $f$ . We assume that  $f$  is not bounded any more but instead, has a linear growth condition. It is the main novelty of this work. To our knowledge, this general case has not been studied in the literature. Finally, we show the existence of Nash equilibrium point for this game. We provide a link between the game which we constructed and BSDE. The existence of the NEP is equivalent to the existence of solutions for a related BSDE, which is multiple-dimensional with continuous generator involving both stochastic linear growth and quadratic terms of  $z$ . The difference with [5] is that the linear term of  $z$  is of linear growth  $\omega$  by  $\omega$  due to the linear growth of  $f$ . Under the generalized Isaacs hypothesis and domination property of solutions for (1.1), which holds when the uniform elliptic condition on  $\sigma$  is satisfied, we show that the associated BSDE has a solution which then provides the NEP for our game.

The paper is organized as follows:

In Section 2, we present the precise model of risk-sensitive nonzero-sum stochastic differential game and necessary hypotheses on related coefficients. In Section 3, we firstly state some useful lemmas. Particularly, Lemma 3.2 and Corollary 3.1, which corresponding to the integrability of Doléan-Dade exponential local martingale, play a crucial role. Then, the link between game and BSDE is demonstrated by Proposition 3.1. The utility function is characterized by the initial value of a BSDE. Then, it turns out by Theorem 3.1 that the existence of the NEP for this game problem is equivalent to the existence of some specific BSDE which is multiple dimensional, with continuous generator involving a quadratic term and a linear growth term of  $Z$ ,  $\omega$  by  $\omega$ . Finally, we show, in Section 4, the solutions for this specific BSDE exist when the generalized Isaacs condition is fulfilled and the law of the dynamic of the system satisfies the  $L^q$ -domination condition. The latter condition is naturally holds if the diffusion coefficient  $\sigma$  satisfies the well-known uniform elliptic condition. Our method to deal with this BSDE with non-regular quadratic generator is that we firstly cancel the quadratic term by applying the exponential transform, then, we take an approximation of the new generator. Besides, in Markovian framework, those approximate processes can be expressed via some deterministic functions. We then provide uniform estimates of the processes, as well as the growth properties of the corresponding deterministic functions. Later, the convergence result is proved. At the end, by taking the inverse transform, the proof for the existence is finished.

## 2 Statement of the risk-sensitive game

In this section, we will give some basic notations, the preliminary assumptions throughout this paper, as well as the statement of the risk-sensitive nonzero-sum stochastic differential game. Let  $(\Omega, \mathcal{F}, \mathbf{P})$  be a probability space on which we define a  $d$ -dimensional Brownian motion  $B = (B_t)_{0 \leq t \leq T}$  with integer  $d \geq 1$  and fixed  $T > 0$ . Let us denote by  $\mathbf{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ , the natural filtration generated by process  $B$  and augmented by  $\mathcal{N}_{\mathbf{P}}$  the  $\mathbf{P}$ -null sets, *i.e.*  $\mathcal{F}_t = \sigma\{B_s, s \leq t\} \vee \mathcal{N}_{\mathbf{P}}$ .

Let  $\mathcal{P}$  be the  $\sigma$ -algebra on  $[0, T] \times \Omega$  of  $\mathcal{F}_t$ -progressively measurable sets. Let  $p \in [1, \infty)$  be real constant and  $t \in [0, T]$  be fixed. We then define the following spaces:

- $\mathcal{L}^p = \{\xi : \mathcal{F}_t\text{-measurable and } \mathbf{R}^m\text{-valued random variable such that } \mathbf{E}[|\xi|^p] < \infty\};$
- $\mathcal{S}_{t,T}^p(\mathbf{R}^m) = \{\varphi = (\varphi_s)_{t \leq s \leq T} : \mathcal{P}\text{-measurable, continuous and } \mathbf{R}^m\text{-valued such that } \mathbf{E}[\sup_{s \in [t, T]} |\varphi_s|^p] < \infty\};$
- $\mathcal{H}_{t,T}^p(\mathbf{R}^m) = \{\varphi = (\varphi_s)_{t \leq s \leq T} : \mathcal{P}\text{-measurable and } \mathbf{R}^m\text{-valued such that } \mathbf{E}[(\int_t^T |\varphi_s|^2 ds)^{\frac{p}{2}}] < \infty\};$
- $\mathcal{D}_{t,T}^p(\mathbf{R}^m) = \{\varphi = (\varphi_s)_{t \leq s \leq T} : \mathcal{P}\text{-measurable and } \mathbf{R}^m\text{-valued such that } \mathbf{E}[\sup_{s \in [t, T]} e^{p\varphi_s}] < \infty\}.$

Hereafter,  $\mathcal{S}_{0,T}^p(\mathbf{R}^m)$ ,  $\mathcal{H}_{0,T}^p(\mathbf{R}^m)$ ,  $\mathcal{D}_{0,T}^p(\mathbf{R}^m)$  are simply denoted by  $\mathcal{S}_T^p(\mathbf{R}^m)$ ,  $\mathcal{H}_T^p(\mathbf{R}^m)$ ,  $\mathcal{D}_T^p(\mathbf{R}^m)$ . The following assumptions are in force throughout this paper. Let  $\sigma$  be the function defined as:

$$\begin{aligned} \sigma : [0, T] \times \mathbf{R}^m &\longrightarrow \mathbf{R}^{m \times m} \\ (t, x) &\longmapsto \sigma(t, x) \end{aligned}$$

which satisfies the following assumptions:

### Assumptions (A1)

(i)  $\sigma$  is uniformly Lipschitz w.r.t  $x$ . *i.e.* there exists a constant  $C_1$  such that,

$$\forall t \in [0, T], \forall x, x' \in \mathbf{R}^m, \quad |\sigma(t, x) - \sigma(t, x')| \leq C_1 |x - x'|.$$

(ii)  $\sigma$  is invertible and bounded and its inverse is bounded, *i.e.*, there exists a constant  $C_\sigma$  such that

$$\forall (t, x) \in [0, T] \times \mathbf{R}^m, \quad |\sigma(t, x)| + |\sigma^{-1}(t, x)| \leq C_\sigma.$$

### Remark 2.1. Uniform elliptic condition.

Under Assumptions (A1), we can verify that, there exists a real constant  $\epsilon > 0$  such that for any  $(t, x) \in [0, T] \times \mathbf{R}^m$ ,

$$\epsilon \cdot I \leq \sigma(t, x) \cdot \sigma^\top(t, x) \leq \epsilon^{-1} \cdot I \quad (2.1)$$

where  $I$  is the identity matrix of dimension  $m$ .

We consider, in this article the 2-player case. The general multiple players game is a straightforward adaption.

For  $(t, x) \in [0, T] \times \mathbf{R}^m$ , let  $X = (X_s^{t,x})_{s \leq T}$  be the solution of the following stochastic differential equation:

$$\begin{cases} X_s^{t,x} = x + \int_t^s \sigma(u, X_u^{t,x}) dB_u, & s \in [t, T]; \\ X_s^{t,x} = x, & s \in [0, t]. \end{cases} \quad (2.2)$$

Under Assumptions (A1) above, we know such  $X$  exists and is unique (see Karatzas and Shreve, pp.289, 1991[16]). Let us now denote by  $U_1$  and  $U_2$  two compact metric spaces and let  $\mathcal{M}_1$  (resp.  $\mathcal{M}_2$ ) be the set of  $\mathcal{P}$ -measurable processes  $u = (u_t)_{t \leq T}$  (resp.  $v = (v_t)_{t \leq T}$ ) with values in  $U_1$  (resp.  $U_2$ ). We denote by  $\mathcal{M}$  the set  $\mathcal{M}_1 \times \mathcal{M}_2$ , hereafter  $\mathcal{M}$  is called the set of admissible controls. We then introduce two Borelian functions

$$\begin{aligned} f &: [0, T] \times \mathbf{R}^m \times U_1 \times U_2 \longrightarrow \mathbf{R}^m, \\ h_i \text{ (resp. } g^i) &: [0, T] \times \mathbf{R}^m \times U_1 \times U_2 \text{ (resp. } \mathbf{R}^m) \longrightarrow \mathbf{R}, \quad i = 1, 2, \end{aligned}$$

which satisfy:

**Assumptions (A2)**

- (i) for any  $(t, x) \in [0, T] \times \mathbf{R}^m$ ,  $(u, v) \mapsto f(t, x, u, v)$  is continuous on  $U_1 \times U_2$ . Moreover  $f$  is of linear growth w.r.t  $x$ , *i.e.* there exists a constant  $C_f$  such that  $|f(t, x, u, v)| \leq C_f(1 + |x|)$ ,  $\forall (t, x, u, v) \in [0, T] \times \mathbf{R}^m \times U_1 \times U_2$ .
- (ii) for any  $(t, x) \in [0, T] \times \mathbf{R}^m$ ,  $(u, v) \mapsto h_i(t, x, u, v)$  is continuous on  $U_1 \times U_2$ ,  $i = 1, 2$ . Moreover, for  $i = 1, 2$ ,  $h_i$  is of sub-quadratic growth w.r.t  $x$ , *i.e.*, there exist constants  $C_h$  and  $1 < \gamma < 2$  such that  $|h_i(t, x, u, v)| \leq C_h(1 + |x|^\gamma)$ ,  $\forall (t, x, u, v) \in [0, T] \times \mathbf{R}^m \times U_1 \times U_2$ .
- (iii) the functions  $g^i$  are of sub-quadratic growth with respect to  $x$ , *i.e.* there exist constants  $C_g$  and  $1 < \gamma < 2$  such that  $|g^i(x)| \leq C_g(1 + |x|^\gamma)$ ,  $\forall x \in \mathbf{R}^m$ , for  $i=1, 2$ .

For  $(u, v) \in \mathcal{M}$ , let  $\mathbf{P}_{t,x}^{u,v}$  be the measure on  $(\Omega, \mathcal{F})$  defined as follows:

$$d\mathbf{P}_{t,x}^{u,v} = \zeta_T \left( \int_0^\cdot \sigma^{-1}(s, X_s^{t,x}) f(s, X_s^{t,x}, u_s, v_s) dB_s \right) d\mathbf{P}, \quad (2.3)$$

where for any  $(\mathcal{F}_t, \mathbf{P})$ -continuous local martingale  $M = (M_t)_{t \leq T}$ ,

$$\zeta(M) := \left( \exp \left\{ M_t - \frac{1}{2} \langle M \rangle_t \right\} \right)_{t \leq T}, \quad (2.4)$$

where  $\langle \cdot \rangle$  denotes the quadratic variation process. We could deduce from Assumptions (A1), (A2)-(i) on  $\sigma$  and  $f$  that  $\mathbf{P}_{t,x}^{u,v}$  is a probability on  $(\Omega, \mathcal{F})$  (see Appendix A, [5] or [16] pp.200). By Girsanov's theorem (Girsanov, 1960 [11], pp.285-301), the process  $B^{u,v} := (B_s - \int_0^s \sigma^{-1}(r, X_r^{t,x}) f(r, X_r^{t,x}, u_r, v_r) dr)_{s \leq T}$  is a  $(\mathcal{F}_s, \mathbf{P}_{t,x}^{u,v})$ -Brownian motion and  $(X_s^{t,x})_{s \leq T}$  satisfies the following stochastic differential equation:

$$\begin{cases} dX_s^{t,x} = f(s, X_s^{t,x}, u_s, v_s)ds + \sigma(s, X_s^{t,x})dB_s^{u,v}, & s \in [t, T]; \\ X_s^{t,x} = x, & s \in [0, t]. \end{cases} \quad (2.5)$$

As a matter of fact, the process  $(X_s^{t,x})_{s \leq T}$  is not adapted with respect to the filtration generated by the Brownian motion  $(B_s^{u,v})_{s \leq T}$  any more, therefore  $(X_s^{t,x})_{s \leq T}$  is called a weak solution for the SDE (2.5). Now the system is controlled by player 1 (resp. Player 2) with  $u$  (resp.  $v$ ).

Now, let us fix  $(t, x)$  to  $(0, x_0)$ , i.e.,  $(t, x) = (0, x_0)$ . For a general risk preference coefficient  $\theta$ , we define the *costs* (or *payoffs*) of the players for  $(u, v) \in \mathcal{M}$  by:

$$J^i(u, v) = \mathbf{E}_{0, x_0}^{u, v} \left[ e^{\theta \left\{ \int_0^T h_i(s, X_s^{0, x_0}, u_s, v_s) ds + g^i(X_T^{0, x_0}) \right\}} \right], \quad i = 1, 2 \quad (2.6)$$

where  $\mathbf{E}_{0, x_0}^{u, v}(\cdot)$  is the expectation under the probability  $\mathbf{P}_{0, x_0}^{u, v}$ . Hereafter  $\mathbf{E}_{0, x_0}^{u, v}$  (resp.  $\mathbf{P}_{0, x_0}^{u, v}$ ) will be simply denoted by  $\mathbf{E}^{u, v}$  (resp.  $\mathbf{P}^{u, v}$ ). The functions  $h_1$  and  $g^1$  (resp.  $h_2$  and  $g^2$ ) are, respectively, the *instantaneous* and *terminal costs* for player 1 (resp. player 2). The player is called risk-averse (resp. risk-seeking) if  $\theta > 0$  (resp.  $\theta < 0$ ). Since the resolution of the problem is the same in all cases ( $\theta > 0$ ,  $\theta < 0$  or  $\theta = 0$ ), without loss of generality, we assume  $\theta = 1$  in (2.6) for simplicity below.

In this article, the quantity  $J^i(u, v)$  is the cost that player  $i$  ( $i = 1, 2$ ) has to pay for his control on the system. The problem is to find a pair of admissible controls  $(u^*, v^*)$  such that:

$$J^1(u^*, v^*) \leq J^1(u, v^*) \text{ and } J^2(u^*, v^*) \leq J^2(u^*, v), \quad \forall (u, v) \in \mathcal{M}.$$

The control  $(u^*, v^*)$  is called a *Nash equilibrium point* for the risk-sensitive nonzero-sum stochastic differential game which means that each player chooses his best control, while, an equilibrium is a pair of controls, such that, when applied, no player will lower his/her cost by unilaterally changing his/her own control.

Let us introduce now the *Hamiltonian functions* for this game, for  $i = 1, 2$ , by  $H_i : [0, T] \times \mathbf{R}^{2m} \times U_1 \times U_2 \rightarrow \mathbf{R}$ , associate:

$$H_i(t, x, p, u, v) = p\sigma^{-1}(t, x)f(t, x, u, v) + h_i(t, x, u, v). \quad (2.7)$$

Besides, we introduce the following assumptions which will play an important role in the proof of existence of equilibrium point.

### Assumptions (A3)

(i) **Generalized Isaacs condition:** There exist two borelian applications  $u_1^*, u_2^*$  defined on  $[0, T] \times \mathbf{R}^{3m}$ , with values in  $U_1$  and  $U_2$  respectively, such that for any  $(t, x, p, q, u, v) \in [0, T] \times \mathbf{R}^{3m} \times U_1 \times U_2$ , we have:

$$H_1^*(t, x, p, q) = H_1(t, x, p, u_1^*(t, x, p, q), u_2^*(t, x, p, q)) \leq H_1(t, x, p, u, u_2^*(t, x, p, q))$$

and

$$H_2^*(t, x, p, q) = H_2(t, x, q, u_1^*(t, x, p, q), u_2^*(t, x, p, q)) \leq H_2(t, x, q, u_1^*(t, x, p, q), v).$$

(ii) The mapping  $(p, q) \in \mathbf{R}^{2m} \mapsto (H_1^*, H_2^*)(t, x, p, q) \in \mathbf{R}$  is continuous for any fixed  $(t, x) \in [0, T] \times \mathbf{R}^m$ .  $\square$

To solve this risk-sensitive stochastic differential game, we adopt the BSDE approach. Precisely speaking, to show the game has a Nash equilibrium point, it is enough to show that its associated BSDE, which is multi-dimensional and with a generator not standard, has a solution (see Theorem 3.1 below). Therefore the main objective of the next section is to study the connection between the risk-sensitive stochastic differential game and BSDEs.

### 3 Risk-sensitive nonzero-sum stochastic differential game and BSDEs

Let  $(t, x) \in [0, T] \times \mathbf{R}^m$  and  $(\theta_s^{t,x})_{s \leq T}$  be the solution of the following forward stochastic differential equation:

$$\begin{cases} d\theta_s = b(s, \theta_s)ds + \sigma(s, \theta_s)dB_s, & s \in [t, T]; \\ \theta_s = x, & s \in [0, t], \end{cases} \quad (3.1)$$

where  $\sigma : [0, T] \times \mathbf{R}^m \rightarrow \mathbf{R}^{m \times m}$  satisfies Assumptions (A1)(i)-(ii) and  $b : [0, T] \times \mathbf{R}^m \rightarrow \mathbf{R}^m$  is a measurable function which verifies the following assumption:

**Assumption (A4):** The function  $b$  is uniformly Lipschitz and bounded, *i.e.*, there exist constants  $C_2$  and  $C_b$  such that:

$$\forall t \in [0, T], \forall x, x' \in \mathbf{R}^m, |b(t, x) - b(t, x')| \leq C_2 |x - x'| \text{ and } |b(t, x)| \leq C_b.$$

Before proceeding further, let us give some useful properties of stochastic process  $(\theta_s^{t,x})_{s \leq T}$ .

**Lemma 3.1.** *Under Assumptions (A1) and (A4), we have*

(i) *the stochastic process  $(\theta_s^{t,x})_{s \leq T}$  has moment of any order, *i.e.* there exists a constant  $C_q \in \mathbf{R}$  such that:  $\mathbf{P}$ -a.s.*

$$\forall q \in [1, \infty), \mathbf{E} \left[ \left( \sup_{s \leq T} |\theta_s^{t,x}| \right)^{2q} \right] \leq C_q (1 + |x|^{2q}); \quad (3.2)$$

(ii) *additionally, it satisfies the following estimate: there exists a constant  $C_{\lambda,l} \in \mathbf{R}$ , such that  $\mathbf{P}$ -a.s.*

$$\forall l \in [1, 2), \lambda \in (0, \infty), \mathbf{E} \left[ e^{\lambda \sup_{s \leq T} |\theta_s^{t,x}|^l} \right] \leq e^{C_{\lambda,l}(1+|x|^l)}. \quad (3.3)$$

Apart from  $q$ ,  $\lambda$  and  $l$ , the constants  $C_q$  and  $C_{\lambda,l}$  in (3.2)(3.3) depend also on  $C_b$  and  $C_\sigma$  and  $T$ .

*Proof.* We refer readers [16] (pp.306) for the result (i). In the following, we only provide the proof of (ii). We denote  $b(s, \theta_s^{t,x})$  and  $\sigma(s, \theta_s^{t,x})$  simply by  $b_s$  and  $\sigma_s$ . Considering  $(b_s)_{s \leq T}$  is bounded and  $\mathbf{E}[f] = \int_0^\infty \mathbf{P}\{f > u\} du$  for all positive function  $f$ , we obtain,

$$\begin{aligned}
& \mathbf{E}[e^{\lambda \sup_{s \leq T} |\theta_s^{t,x}|^l}] \\
&= \mathbf{E}[e^{\lambda \sup_{s \leq T} |x + \int_t^s b_r ds + \int_t^s \sigma_r dB_r|^l}] \\
&\leq e^{C_{l,\lambda,b,T} \cdot (1+|x|^l)} \mathbf{E}[e^{C_{l,\lambda} \cdot \sup_{s \leq T} |\int_0^s \sigma_r dB_r|^l}] \\
&= e^{C_{l,\lambda,b,T} \cdot (1+|x|^l)} \int_0^\infty \mathbf{P}\{e^{C_{l,\lambda} \cdot \sup_{s \leq T} |\int_0^s \sigma_r dB_r|^l} > u\} du \\
&= e^{C_{l,\lambda,b,T} \cdot (1+|x|^l)} \left(1 + \int_1^\infty \mathbf{P}\{e^{C_{l,\lambda} \cdot \sup_{s \leq T} |\int_0^s \sigma_r dB_r|^l} > e^{C_{l,\lambda} \cdot u^l}\} de^{C_{l,\lambda} \cdot u^l}\right) \\
&= e^{C_{l,\lambda,b,T} \cdot (1+|x|^l)} \left(1 + \int_0^\infty \mathbf{P}\{\sup_{s \leq T} |\int_0^s \sigma_r dB_r| > u\} e^{C_{l,\lambda} \cdot u^l} C_{l,\lambda} l u^{l-1} du\right).
\end{aligned}$$

Apply Theorem 2 in [4] (pp.247),  $\mathbf{P}\{\sup_{s \leq T} |\int_0^s \sigma_r dB_r| > u\} \leq e^{-\frac{u^2}{2TC_\sigma^2}}$ . Therefore,

$$\begin{aligned}
& \mathbf{E}[e^{\lambda \sup_{s \leq T} |\theta_s^{t,x}|^l}] \\
&\leq e^{C_{l,\lambda,b,T} \cdot (1+|x|^l)} \left(1 + \int_0^\infty e^{-\frac{u^2}{2TC_\sigma^2}} e^{C_{l,\lambda} \cdot u^l} C_{l,\lambda} l u^{l-1} du\right) \\
&\leq e^{C_{l,\lambda,b,T} \cdot (1+|x|^l)}.
\end{aligned}$$

The above inequality is finite since  $1 \leq l < 2$  and  $u \leq e^u$  for any  $u > 0$ .  $\square$

Next let us recall the following result by Hausmann ([14], pp.14) related to integrability of the Doléan-Dade exponential local martingale defined by (2.4).

**Lemma 3.2.** Assume (A1)-(i)(ii) and (A4), let  $(\theta_s^{t,x})_{s \leq T}$  be the solution of (3.1) and  $\varphi$  be a  $\mathcal{P} \otimes \mathcal{B}(\mathbf{R}^m)$ -measurable application from  $[0, T] \times \Omega \times \mathbf{R}^m$  to  $\mathbf{R}^m$  which is of linear growth, that is,  $\mathbf{P}$ -a.s.,  $\forall (s, x) \in [0, T] \times \mathbf{R}^m$ ,

$$|\varphi(s, \omega, x)| \leq C_3(1 + |x|).$$

Then, there exists some  $p \in (1, 2)$  and a constant  $C$ , where  $p$  depends only on  $C_\sigma$ ,  $C_2$ ,  $C_b$ ,  $C_3$ ,  $m$  while the constant  $C$ , depends only on  $m$  and  $p$ , but not on  $\varphi$ , such that:

$$\mathbf{E} \left[ \left| \zeta_T \left( \int_0^\cdot \varphi(s, \theta_s^{t,x}) dB_s \right) \right|^p \right] \leq C, \quad (3.4)$$

where the process  $\zeta(\int_0^\cdot \varphi(s, \theta_s^{t,x}) dB_s)$  is the density function defined in (2.4).

It follows from Lemma 3.2 that,

**Corollary 3.1.** For an admissible control  $(u, v) \in \mathcal{M}$  and  $(t, x) \in [0, T] \times \mathbf{R}^m$ , there exists some  $p_0 \in (1, 2)$  and a constant  $C$ , such that

$$\mathbf{E} \left[ \left| \zeta_T \left( \int_0^\cdot \sigma(s, X_s^{t,x})^{-1} f(s, X_s^{t,x}, u_s, v_s) dB_s \right) \right|^{p_0} \right] \leq C. \quad (3.5)$$



**Remark 3.1.** Corollary 3.1 is needed for us in the proofs of Proposition 3.1 and Theorem 3.1 which is the main result of this work. Notice that the function  $f$  is no longer bounded as in the literature but is of linear growth in  $x$ .

As a by-product of Lemma 3.1 and 3.2, we also have the similar estimates for the process  $X^{t,x}$ .

**Lemma 3.3.** (i) There exist two constants  $\bar{C}_q, \bar{C}_{\lambda,l} \in \mathbf{R}$ , such that  $\mathbf{P}$ -a.s.

$$\forall q \in [1, \infty), \mathbf{E} \left[ \left( \sup_{s \leq T} |X_s^{t,x}| \right)^{2q} \right] \leq \bar{C}_q (1 + |x|^{2q}), \quad (3.6)$$

and

$$\forall l \in [1, 2), \lambda \in (0, \infty), \mathbf{E} \left[ e^{\lambda \sup_{s \leq T} |X_s^{t,x}|^l} \right] \leq e^{\bar{C}_{\lambda,l}(1+|x|^l)}; \quad (3.7)$$

(ii) Moreover, for solutions of the weak formulation of SDEs (2.5), we have the similar results. Precisely speaking, for  $(u, v) \in \mathcal{M}$ ,  $\mathbf{E}_{t,x}^{u,v}$  is the expectation under the probability  $\mathbf{P}_{t,x}^{u,v}$ , then there exist constants  $\tilde{C}_q, \tilde{C}_{\lambda,l} \in \mathbf{R}$ , such that  $\mathbf{P}$ -a.s.

$$\forall q \in [1, \infty), \mathbf{E}_{t,x}^{u,v} \left[ \left( \sup_{s \leq T} |X_s^{t,x}| \right)^{2q} \right] \leq \tilde{C}_q (1 + |x|^{2q}), \quad (3.8)$$

and

$$\forall l \in [1, 2), \lambda \in (0, \infty), \mathbf{E}_{t,x}^{u,v} \left[ e^{\lambda \sup_{s \leq T} |X_s^{t,x}|^l} \right] \leq e^{\tilde{C}_{\lambda,l}(1+|x|^l)}. \quad (3.9)$$

*Proof.* We only prove (3.9). Since,

$$\mathbf{E}_{t,x}^{u,v} \left[ e^{\lambda \sup_{s \leq T} |X_s^{t,x}|^l} \right] = \mathbf{E} \left[ e^{\lambda \sup_{s \leq T} |X_s^{t,x}|^l} \cdot \zeta_T \right],$$

where  $\zeta_T$  represents  $\zeta_T(\int_0^\cdot \sigma(s, X_s^{t,x})^{-1} f(s, X_s^{t,x}, u_s, v_s) dB_s)$ . As a result of Corollary 3.1, there exists some  $p_0 \in (1, 2)$ , such that,  $\zeta_T \in L^{p_0}$ . Therefore, by Young's inequality and (3.7), we obtain that,

$$\begin{aligned} \mathbf{E}_{t,x}^{u,v} \left[ e^{\lambda \sup_{s \leq T} |X_s^{t,x}|^l} \right] &\leq \mathbf{E} \left[ e^{\frac{p_0 \lambda}{p_0 - 1} \sup_{s \leq T} |X_s^{t,x}|^l} \right] + \mathbf{E} [|\zeta_T|^{p_0}] \\ &\leq e^{\bar{C}_{\lambda,l,p_0}(1+|x|^l)} + C_{m,p_0} \\ &\leq e^{\tilde{C}_{\lambda,l,m,p_0}(1+|x|^l)}. \end{aligned}$$

□

The next proposition characterizes the payoff function  $J^i(u, v)$  for  $i = 1, 2$  with form (2.6) by means of BSDEs. It turns out that the payoffs  $J^i(u, v)$  can be expressed as the exponential of the initial value for a related BSDE. It is multidimensional, with a continuous generator involving a quadratic term of  $Z$ .

**Proposition 3.1.** Under Assumptions (A1) and (A2), for any admissible control  $(u, v) \in \mathcal{M}$ , there exists a pair of adapted processes  $(Y^{i,(u,v)}, Z^{i,(u,v)})$ ,  $i = 1, 2$ , with values on  $\mathbf{R} \times \mathbf{R}^m$  such that:

(i) For any  $p > 1$ ,

$$\mathbf{E}^{u,v} \left[ \sup_{0 \leq t \leq T} e^{pY_t^{i,(u,v)}} \right] < \infty \text{ and } \mathbf{P} - a.s. \int_0^T |Z_t^{i,(u,v)}|^2 dt < \infty. \quad (3.10)$$

(ii) For  $t \leq T$ ,

$$\begin{aligned} Y_t^{i,(u,v)} &= g^i(X_T^{0,x_0}) + \int_t^T \left\{ H_i(s, X_s^{0,x_0}, Z_s^{i,(u,v)}, u_s, v_s) + \frac{1}{2} |Z_s^{i,(u,v)}|^2 \right\} ds \\ &\quad - \int_t^T Z_s^{i,(u,v)} dB_s. \end{aligned} \quad (3.11)$$

The solution is unique for fixed  $x_0 \in \mathbf{R}^m$ . Moreover,  $J^i(u, v) = e^{Y_0^{i,(u,v)}}$ .

*Proof. Part I : Existence and uniqueness.* We take the case of  $i = 1$  for example, and of course the case of  $i = 2$  can be solved in a similar way. The main method here is to define a reasonable form of the solution directly. We first eliminate the quadratic term in the generator by applying the classical exponential exchange. Then, the definition of  $Y$  component is closely related to Girsanov's transformation, and the process  $Z$  is given by the martingale representation theorem. Afterwards, we shall verify by Itô's formula that what we defined above is exactly the solution of the original BSDE.

As we stated in the previous section, the process  $(X_s^{0,x_0})_{s \leq T}$  satisfies SDE (2.5) by substituting  $(0, x_0)$  for  $(t, x)$ .

In order to remove the quadratic part in the generator of BSDE (3.11), we first take the classical exponential exchange as follows:  $\forall t \leq T$ , let

$$\begin{cases} \bar{Y}_t^{1,(u,v)} = e^{Y_t^{1,(u,v)}}; \\ \bar{Z}_t^{1,(u,v)} = \bar{Y}_t^{1,(u,v)} Z_t^{1,(u,v)}. \end{cases}$$

Therefore, the processes  $(\bar{Y}_t^{1,(u,v)}, \bar{Z}_t^{1,(u,v)})_{t \leq T}$  solve the following BSDE:

$$\begin{aligned} \bar{Y}_t^{1,(u,v)} &= e^{g^1(X_T^{0,x_0})} + \int_t^T \bar{Z}_s^{1,(u,v)} \sigma^{-1}(s, X_s^{0,x_0}) f(s, X_s^{0,x_0}, u_s, v_s) \\ &\quad + (\bar{Y}_s^{1,(u,v)})^+ h(s, X_s^{0,x_0}, u_s, v_s) ds - \int_t^T \bar{Z}_s^{1,(u,v)} dB_s, \quad t \leq T. \end{aligned} \quad (3.12)$$

Applying Girsanov's transformation as indicated by (2.3)-(2.4), the BSDE (3.12) then reduces to

$$\bar{Y}_t^{1,(u,v)} = e^{g^1(X_T^{0,x_0})} + \int_t^T (\bar{Y}_s^{1,(u,v)})^+ h(s, X_s^{0,x_0}, u_s, v_s) ds - \int_t^T \bar{Z}_s^{1,(u,v)} dB_s^{u,v}, \quad t \leq T.$$

Let us now define the process  $\bar{Y}^{1,(u,v)}$  explicitly by:

$$\bar{Y}_t^{1,(u,v)} := \mathbf{E}^{u,v} \left[ \exp \left\{ g^1(X_T^{0,x_0}) + \int_t^T h_1(s, X_s^{0,x_0}, u_s, v_s) ds \right\} \middle| \mathcal{F}_t \right], \quad t \leq T. \quad (3.13)$$

Considering the sub-quadratic growth Assumptions (A2)-(ii)(iii) on  $h_1$  and  $g^1$  and the estimate (3.9), we obtain,

$$\begin{aligned} \mathbf{E}^{u,v} \left[ \exp \left\{ g^1(X_T^{0,x_0}) + \int_0^T h_1(s, X_s^{0,x_0}, u_s, v_s) ds \right\} \right] \\ \leq \mathbf{E}^{u,v} \left[ \exp \left\{ C \sup_{0 \leq s \leq T} \left( 1 + |X_s^{0,x_0}|^\gamma \right) \right\} \right] < \infty, \end{aligned}$$

with constant  $C = C_g \vee (TC_h)$ . Therefore, we claim that the process  $(\bar{Y}_t^{1,(u,v)})_{t \leq T}$  in (3.13) is well-defined.

We will give now the definition of process  $(\bar{Z}_t^{1,(u,v)})_{t \leq T}$ . In the following, for notation convenience, we denote by  $\zeta$  the following process  $\zeta := (\zeta_t)_{t \leq T} = (\zeta_t(\int_0^\cdot \sigma^{-1}(s, X_s^{0,x_0}) f(s, X_s^{0,x_0}, u_s, v_s) dB_s))_{t \leq T}$ . Then the definition (3.13) can be rewritten as:

$$\bar{Y}_t^{1,(u,v)} = \zeta_t^{-1} \cdot \mathbf{E} \left[ \zeta_T \cdot \exp \left\{ g^1(X_T^{0,x_0}) + \int_t^T h_1(s, X_s^{0,x_0}, u_s, v_s) ds \right\} \middle| \mathcal{F}_t \right], \quad t \leq T. \quad (3.14)$$

Thanks to Corollary 3.1, there exists some  $p_0 \in (1, 2)$ , such that  $\mathbf{E}[|\zeta_T|^{p_0}] < \infty$ . Therefore, from Young's inequality, we get that for any constant  $q \in (1, p_0)$ ,

$$\begin{aligned} \mathbf{E} \left[ \left| \zeta_T \cdot \exp \left\{ g^1(X_T^{0,x_0}) + \int_0^T h_1(s, X_s^{0,x_0}, u_s, v_s) ds \right\} \right|^q \right] \\ \leq \frac{q}{p_0} \mathbf{E} [|\zeta_T|^{p_0}] + \frac{p_0 - q}{p_0} \mathbf{E} \left[ \exp \left\{ \frac{qp_0}{p_0 - q} \cdot \left( g^1(X_T^{0,x_0}) + \int_0^T h_1(s, X_s^{0,x_0}, u_s, v_s) ds \right) \right\} \right]. \end{aligned}$$

As a consequence of Assumptions (A2)-(ii)(iii) and (3.3), the following expectation is finite, i.e.,

$$\frac{p_0 - q}{p_0} \mathbf{E} \left[ \exp \left\{ \frac{qp_0}{p_0 - q} \cdot \left( g^1(X_T^{0,x_0}) + \int_0^T h_1(s, X_s^{0,x_0}, u_s, v_s) ds \right) \right\} \right] < \infty$$

Then, we deduce that,

$$\zeta_T \cdot \exp \left\{ g^1(X_T^{0,x_0}) + \int_0^T h_1(s, X_s^{0,x_0}, u_s, v_s) ds \right\} \in \mathcal{L}^q(d\mathbf{P}).$$

It follows from (3.14) and the representation theorem that, there exists a  $\mathcal{P}$ -measurable process  $(\bar{\theta}_s)_{s \leq T} \in \mathcal{H}_T^q(\mathbf{R}^m)$ , such that for any  $t \leq T$ ,

$$\begin{aligned} \bar{Y}_t^{1,(u,v)} &= \zeta_t^{-1} \exp \left\{ - \int_0^t h_1(s, X_s^{0,x_0}, u_s, v_s) ds \right\} \times \\ &\quad \times \left\{ \mathbf{E} \left[ \zeta_T \exp \left\{ g^1(X_T^{0,x_0}) + \int_0^T h_1(s, X_s^{0,x_0}, u_s, v_s) ds \right\} \right] + \int_0^t \bar{\theta}_s dB_s \right\} \end{aligned}$$

Let us denote by:

$$R_t := \mathbf{E} \left[ \zeta_T \exp \left\{ g^1(X_T^{0,x_0}) + \int_0^T h_1(s, X_s^{0,x_0}, u_s, v_s) ds \right\} \right] + \int_0^t \bar{\theta}_s dB_s, \quad t \leq T.$$

Taking account of  $d\zeta_t = \zeta_t \sigma^{-1}(t, X_t^{0,x_0}) f(t, X_t^{0,x_0}, u_t, v_t) dB_t$  for  $t \leq T$ , then by Itô's formula, we have  $d\zeta_t^{-1} = -\zeta_t^{-1} \{ \sigma^{-1}(t, X_t^{0,x_0}) f(t, X_t^{0,x_0}, u_t, v_t) dB_t - |\sigma^{-1}(t, X_t^{0,x_0}) f(t, X_t^{0,x_0}, u_t, v_t)|^2 dt \}$ ,  $t \leq T$ . Moreover,

$$\begin{aligned} & d \left[ \zeta_t^{-1} \exp \left\{ - \int_0^t h_1(s, X_s^{0,x_0}, u_s, v_s) ds \right\} \right] \\ &= -\zeta_t^{-1} \exp \left\{ - \int_0^t h_1(s, X_s^{0,x_0}, u_s, v_s) ds \right\} \left\{ \sigma^{-1}(t, X_t^{0,x_0}) f(t, X_t^{0,x_0}, u_t, v_t) dB_t \right. \\ &\quad \left. + \left[ -|\sigma^{-1}(t, X_t^{0,x_0}) f(t, X_t^{0,x_0}, u_t, v_t)|^2 + h_1(t, X_t^{0,x_0}, u_t, v_t) \right] dt \right\}, \quad t \leq T. \end{aligned}$$

Hence, for  $t \leq T$ ,

$$\begin{aligned} d\bar{Y}_t^{1,(u,v)} &= -\zeta_t^{-1} \exp \left\{ - \int_0^t h_1(s, X_s^{0,x_0}, u_s, v_s) ds \right\} \left\{ \sigma^{-1}(t, X_t^{0,x_0}) f(t, X_t^{0,x_0}, u_t, v_t) dB_t \right. \\ &\quad \left. + \left[ -|\sigma^{-1}(t, X_t^{0,x_0}) f(t, X_t^{0,x_0}, u_t, v_t)|^2 + h_1(t, X_t^{0,x_0}, u_t, v_t) \right] dt \right\} R_t \\ &\quad + \zeta_t^{-1} \exp \left\{ - \int_0^t h_1(s, X_s^{0,x_0}, u_s, v_s) ds \right\} \bar{\theta}_t dB_t \\ &\quad - \zeta_t^{-1} \exp \left\{ - \int_0^t h_1(s, X_s^{0,x_0}, u_s, v_s) ds \right\} \sigma^{-1}(t, X_t^{0,x_0}) f(t, X_t^{0,x_0}, u_t, v_t) \bar{\theta}_t dt, \end{aligned}$$

which allows us to define the process  $\bar{Z}^{1,(u,v)}$  as the volatility coefficient of the above equation, *i.e.*, for  $t \leq T$ ,

$$\begin{aligned} \bar{Z}_t^{1,(u,v)} &:= -\zeta_t^{-1} \exp \left\{ - \int_0^t h_1(s, X_s^{0,x_0}, u_s, v_s) ds \right\} \left\{ \sigma^{-1}(t, X_t^{0,x_0}) f(t, X_t^{0,x_0}, u_t, v_t) R_t \right. \\ &\quad \left. - \bar{\theta}_t \right\}. \end{aligned} \quad (3.15)$$

Then, it is not difficult to verify that the process  $(\bar{Y}_t^{1,(u,v)}, \bar{Z}_t^{1,(u,v)})_{t \leq T}$ , as we defined by (3.13) (3.15) satisfies the BSDE (3.12). Moreover, it can be seen obviously from (3.13) that  $\bar{Y}_t^{1,(u,v)} > 0$  for all  $t \in [0, T]$ . Therefore, we define the pair of processes  $(Y^{1,(u,v)}, Z^{1,(u,v)})$  as follows:

$$\begin{cases} Y_t^{1,(u,v)} = \ln \bar{Y}_t^{1,(u,v)}; \\ Z_t^{1,(u,v)} = \frac{\bar{Z}_t^{1,(u,v)}}{\bar{Y}_t^{1,(u,v)}}, \quad t \leq T. \end{cases}$$

which completes the proof of existence.

The uniqueness is natural by the above construction itself for fixed  $x_0 \in \mathbf{R}^m$ . Since, the solution of BSDE (3.12), if exists, will be of the form (3.13) and (3.15).

*Part II : Norm estimates.* Finally, let us focus on the estimate of  $(Y_t^{1,(u,v)})_{t \leq T}$  which is needed in the next theorem. First, as a consequence of the definition (3.13)

that for any  $p > 1$ ,

$$\begin{aligned} & \mathbf{E}^{u,v} \left[ \left| \sup_{t \in [0,T]} \bar{Y}_t^{1,(u,v)} \right|^p \right] \\ & \leq \mathbf{E}^{u,v} \left[ \left| \sup_{t \in [0,T]} \left\{ \mathbf{E}^{u,v} \left[ \exp \{ g^1(X_T^{0,x_0}) + \int_0^T |h_1(s, X_s^{0,x_0}, u_s, v_s)| ds \} | \mathcal{F}_t \right] \right\} \right|^p \right]. \end{aligned} \quad (3.16)$$

Noticing that the process  $\mathbf{E}^{u,v} \left[ \exp \{ g^1(X_T^{0,x_0}) + \int_0^T |h_1(s, X_s^{0,x_0}, u_s, v_s)| ds \} | \mathcal{F}_t \right]$  is a  $\mathcal{F}_t$ -martingale, then Doob's maximal inequality (see [16] pp.14) implies that,

$$\begin{aligned} & \mathbf{E}^{u,v} \left[ \left| \sup_{t \in [0,T]} \bar{Y}_t^{1,(u,v)} \right|^p \right] \\ & \leq \left( \frac{p}{p-1} \right)^p \mathbf{E}^{u,v} \left[ \left| \mathbf{E}^{u,v} \left[ \exp \{ g^1(X_T^{0,x_0}) + \int_0^T |h_1(s, X_s^{0,x_0}, u_s, v_s)| ds \} | \mathcal{F}_T \right] \right|^p \right] \end{aligned} \quad (3.17)$$

Then, considering the Jensen's inequality and Assumption (A2)(ii)-(iii) on  $g^1$  and  $h_1$ , it turns out that,

$$\begin{aligned} & \mathbf{E}^{u,v} \left[ \left| \sup_{t \in [0,T]} \bar{Y}_t^{1,(u,v)} \right|^p \right] \\ & \leq \left( \frac{p}{p-1} \right)^p \mathbf{E}^{u,v} \left[ \exp \{ p g^1(X_T^{0,x_0}) + p \int_0^T |h_1(s, X_s^{0,x_0}, u_s, v_s)| ds \} \right] \quad (3.18) \\ & \leq \left( \frac{p}{p-1} \right)^p \mathbf{E}^{u,v} \left[ e^{\sup_{t \in [0,T]} C(1+|X_t^{0,x_0}|^\gamma)} \right] < \infty, \end{aligned}$$

which is given by the estimate (3.9) with constant  $C$  depending on  $p$ ,  $C_g$ ,  $C_h$ , and  $T$ . Therefore,

$$\mathbf{E}^{u,v} \left[ \sup_{t \in [0,T]} |\bar{Y}_t^{1,(u,v)}|^p \right] < \infty, \quad (3.19)$$

which gives,

$$\mathbf{E}^{u,v} \left[ \sup_{t \in [0,T]} e^{p Y_t^{1,(u,v)}} \right] < \infty, \quad \forall p > 1.$$

At last, note that in taking  $t = 0$  in (3.13) we obtain  $J^1(u, v) = \bar{Y}_0^{1,(u,v)} = e^{Y_0^{1,(u,v)}}$  since  $\mathcal{F}_0$  contains only  $\mathbf{P}$  and  $\mathbf{P}^{u,v}$  null sets.  $\square$

We are now ready to demonstrate the existence of Nash equilibrium point which is the main result of this article.

**Theorem 3.1.** *Let us assume that:*

- (i) *Assumptions (A1), (A2) and (A3) are fulfilled ;*
- (ii) *There exist two pairs of  $\mathcal{P}$ -measurable processes  $(Y^i, Z^i)$  with values in  $\mathbf{R}^{1+m}$ ,  $i = 1, 2$ , and two deterministic functions  $\varpi^i(t, x)$  which are of subquadratic growth,*

i.e.  $|\varpi^i(t, x)| \leq C(1 + |x|^\gamma)$  with  $1 < \gamma < 2$ ,  $i = 1, 2$  such that:

$$\left\{ \begin{array}{l} \mathbf{P}\text{-a.s.}, \forall t \leq T, Y_t^i = \varpi^i(t, X_t^{0,x}) \text{ and } Z^i \text{ is } dt\text{-square integrable } \mathbf{P}\text{-a.s.}; \\ Y_t^i = g^i(X_T^{0,x}) + \int_t^T \{H_i(s, X_s^{0,x}, Z_s^i, (u^*, v^*)(s, X_s^{0,x}, Z_s^1, Z_s^2)) + \frac{1}{2} |Z_s^i|^2\} ds \\ \quad - \int_t^T Z_s^i dB_s, \quad \forall t \leq T. \end{array} \right. \quad (3.20)$$

Then the pair of control  $(u^*(s, X_s^{0,x}, Z_s^1, Z_s^2), v^*(s, X_s^{0,x}, Z_s^1, Z_s^2))_{s \leq T}$  is admissible and a Nash equilibrium point for the game.

*Proof.* For  $s \leq T$ , let us set  $u_s^* = u^*(s, X_s^{0,x}, Z_s^1, Z_s^2)$  and  $v_s^* = v^*(s, X_s^{0,x}, Z_s^1, Z_s^2)$ . Then  $(u^*, v^*) \in \mathcal{M}$ . On the other hand, we obviously have  $J^1(u^*, v^*) = e^{Y_0^1}$  by Proposition 3.1. Next for an arbitrary element  $u \in \mathcal{M}_1$ , let us show that  $e^{Y^1} \leq e^{Y^{u, v^*}}$ , which yields  $e^{Y_0^1} = J^1(u^*, v^*) \leq J^1(u, v^*) = e^{Y_0^{1, (u, v^*)}}$ . We focus on this point below. For the admissible control  $(u, v^*)$ , thanks to Proposition 3.1, there exists a pair of  $\mathcal{P}$ -measurable processes  $(Y_t^{i, (u, v^*)}, Z_t^{i, (u, v^*)})_{t \leq T}$  for  $i = 1, 2$ , which satisfies: for any  $p > 1$ ,

$$\left\{ \begin{array}{l} Y^{i, (u, v^*)} \in \mathcal{D}_T^p(\mathbf{R}, d\mathbf{P}^{u, v^*}), \quad Z^{i, (u, v^*)} \text{ is } dt\text{-square integrable } \mathbf{P}\text{-a.s.} \\ Y_t^{i, (u, v^*)} = g^i(X_T^{0,x}) + \int_t^T \{H_i(s, X_s^{0,x}, Z_s^{i, (u, v^*)}, u_s, v_s^*) + \frac{1}{2} |Z_s^{i, (u, v^*)}|^2\} dt \\ \quad - \int_t^T Z_s^{i, (u, v^*)} dB_s, \quad \forall t \leq T. \end{array} \right. \quad (3.21)$$

Let us set:  $\forall t \leq T$ ,

$$D_t^{u^*, v^*} := e^{Y_t^1}, \quad D_t^{u, v^*} := e^{Y_t^{1, (u, v^*)}}.$$

Thus Itô-Meyer formula yields, for any  $t \leq T$ ,

$$\begin{aligned} & -d(D_t^{u^*, v^*} - D_t^{u, v^*})^+ + dL_t^0(D_t^{u^*, v^*} - D_t^{u, v^*}) \\ &= \left[ D_t^{u^*, v^*} H_1(t, X_t^{0,x}, Z_t^1, u_t^*, v_t^*) - D_t^{u, v^*} H_1(t, X_t^{0,x}, Z_t^{1, (u, v^*)}, u_t, v_t^*) \right] 1_{\{D_t^{u^*, v^*} - D_t^{u, v^*} > 0\}} dt \\ & \quad - (D_t^{u^*, v^*} Z_t^1 - D_t^{u, v^*} Z_t^{1, (u, v^*)}) 1_{\{D_t^{u^*, v^*} - D_t^{u, v^*} > 0\}} dB_t \\ &= \left[ D_t^{u^*, v^*} H_1(t, X_t^{0,x}, Z_t^1, u_t^*, v_t^*) - D_t^{u^*, v^*} H_1(t, X_t^{0,x}, Z_t^1, u_t, v_t^*) \right. \\ & \quad \left. + D_t^{u^*, v^*} H_1(t, X_t^{0,x}, Z_t^1, u_t, v_t^*) - D_t^{u, v^*} H_1(t, X_t^{0,x}, Z_t^{1, (u, v^*)}, u_t, v_t^*) \right] 1_{\{D_t^{u^*, v^*} - D_t^{u, v^*} > 0\}} dt \\ & \quad - (D_t^{u^*, v^*} Z_t^1 - D_t^{u, v^*} Z_t^{1, (u, v^*)}) 1_{\{D_t^{u^*, v^*} - D_t^{u, v^*} > 0\}} dB_t \\ &= \left[ D_t^{u^*, v^*} \left( H_1(t, X_t^{0,x}, Z_t^1, u_t^*, v_t^*) - H_1(t, X_t^{0,x}, Z_t^1, u_t, v_t^*) \right) \right. \\ & \quad \left. + (D_t^{u^*, v^*} - D_t^{u, v^*})^+ h_1(t, X_t^{0,x}, u_t, v_t^*) \right. \\ & \quad \left. + (D_t^{u^*, v^*} Z_t^1 - D_t^{u, v^*} Z_t^{1, (u, v^*)}) \sigma^{-1}(t, X_t^{0,x}) f(t, X_t^{0,x}, u_t, v_t^*) \right] 1_{\{D_t^{u^*, v^*} - D_t^{u, v^*} > 0\}} dt \\ & \quad - (D_t^{u^*, v^*} Z_t^1 - D_t^{u, v^*} Z_t^{1, (u, v^*)}) 1_{\{D_t^{u^*, v^*} - D_t^{u, v^*} > 0\}} dB_t, \end{aligned} \quad (3.22)$$

where  $L_t^0 = L_t^0(D^{u^*,v^*} - D^{u,v^*})$  is the local time of the continuous semimartingale  $D^{u^*,v^*} - D^{u,v^*}$  at time 0. Next for  $t \leq T$ , let us give  $B_t^{u,v^*} = (B_t - \int_0^t \sigma^{-1}(s, X_s^{0,x}) f(s, X_s^{0,x}, u_s, v_s^*) ds)_{t \leq T}$  which is an  $\mathcal{F}_t$ -Brownian motion under the probability  $\mathbf{P}^{u,v^*}$ , whose density w.r.t.  $\mathbf{P}$  is defined by  $\zeta_T := \zeta_T(\int_0^T \sigma^{-1}(s, X_s^{0,x}) f(s, X_s^{0,x}, u_s, v_s^*) dB_s)$  as defined in (2.3). On the other hand, let us denote:

$$\Gamma_t^1 := (D_t^{u^*,v^*} - D_t^{u,v^*})^+ \exp\left\{\int_0^t h_1(s, X_s^{0,x}, u_s, v_s^*) ds\right\}, \quad t \leq T.$$

Taking into account of (3.22), we then conclude by Itô's formula and Girsanov's transformation that, for  $t \leq T$ ,

$$\begin{aligned} d\Gamma_t^1 &= -\exp\left\{\int_0^t h_1(s, X_s^{0,x}, u_s, v_s^*) ds\right\} \times \\ &\quad \times \left[ D_t^{u^*,v^*} \Delta_t^1 dt - (D_t^{u^*,v^*} Z_t^1 - D_t^{u,v^*} Z_t^{1,(u,v^*)}) dB_t^{u,v^*} - dL_t^0 \right], \end{aligned} \quad (3.23)$$

where

$$\Delta_t^1 = H_1(t, X_t^{0,x}, Z_t^1, u_t^*, v_t^*) - H_1(t, X_t^{0,x}, Z_t^1, u_t, v_t^*) \leq 0,$$

which is obtained by the generalized Isaacs' Assumption (A3)-(i). Next, let us define the stopping time  $\tau_n$  as follows:

$$\tau_n = \inf\{t \geq 0, |D_t^{u,v^*}| + |D_t^{u^*,v^*}| + \int_0^t (|Z_s^1|^2 + |Z_s^{1,(u,v^*)}|^2) ds \geq n\} \wedge T.$$

The sequence of stopping times  $(\tau_n)_{n \geq 0}$  is of stationary type and converges to  $T$  as  $n \rightarrow \infty$ . We then claim that,  $\int_0^{t \wedge \tau_n} \exp\left\{\int_0^s h_1(r, X_r^{0,x}, u_r, v_r^*) dr\right\} 1_{\{D_s^{u^*,v^*} - D_s^{u,v^*} > 0\}} (D_s^{u^*,v^*} Z_s^1 - D_s^{u,v^*} Z_s^{1,(u,v^*)}) dB_s^{u,v^*}$  is a  $\mathcal{F}_t$ -martingale under the probability  $\mathbf{P}^{u,v^*}$  as the following expectation

$$\begin{aligned} &\mathbf{E}^{u,v^*} \left[ \int_0^{\tau_n} e^{2 \int_0^s h_1(r, X_r^{0,x}, u_r, v_r^*) dr} (D_s^{u^*,v^*} Z_s^1 - D_s^{u,v^*} Z_s^{1,(u,v^*)})^2 ds \right] \\ &\leq \mathbf{E}^{u,v^*} \left[ \int_0^{\tau_n} e^{2 \int_0^s h_1(r, X_r^{0,x}, u_r, v_r^*) dr} \left( 2|D_s^{u^*,v^*}|^2 |Z_s^1|^2 + 2|D_s^{u,v^*}|^2 |Z_s^{1,(u,v^*)}|^2 \right) ds \right] \\ &\leq \mathbf{E}^{u,v^*} \left[ \sup_{0 \leq s \leq \tau_n} \left\{ 2e^{2C_h(1+|X_s^{0,x}|^\gamma)} |D_s^{u^*,v^*}|^2 \right\} \cdot \int_0^{\tau_n} |Z_s^1|^2 ds \right] + \\ &\quad \mathbf{E}^{u,v^*} \left[ \sup_{0 \leq s \leq \tau_n} \left\{ 2e^{2C_h(1+|X_s^{0,x}|^\gamma)} |D_s^{u,v^*}|^2 \right\} \cdot \int_0^{\tau_n} |Z_s^{1,(u,v^*)}|^2 ds \right] \end{aligned} \quad (3.24)$$

is finite which is the consequence of the definition of  $\tau_n$  and the estimate (3.9). Considering that  $L_t^0$  is an increasing process, therefore,  $\int_{t \wedge \tau_n}^{\tau_n} \exp\left\{\int_0^s h_1(r, X_r^{0,x}, u_r, v_r^*) dr\right\} dL_s^0$  is positive. Now returning to equation (3.23), then taking integral on interval  $(t \wedge \tau_n, \tau_n)$  and conditional expectation w.r.t.  $\mathcal{F}_{t \wedge \tau_n}$  under the probability  $\mathbf{P}^{u,v^*}$ , yield that,

$$\Gamma_{t \wedge \tau_n}^1 \leq \mathbf{E}^{u,v^*} \left[ \Gamma_{\tau_n}^1 \middle| \mathcal{F}_{t \wedge \tau_n} \right],$$

i.e.,

$$\mathbf{E}^{u,v^*} \Gamma_{t \wedge \tau_n}^1 \leq \mathbf{E}^{u,v^*} \Gamma_{\tau_n}^1. \quad (3.25)$$

Indeed, for any  $p > 1$ ,  $1 < q < p$ , and given  $1 < \gamma < 2$ , we have,

$$\begin{aligned} & \mathbf{E}^{u,v^*} \left[ \sup_{0 \leq t \leq T} |\Gamma_t^1|^q \right] \\ &= \mathbf{E}^{u,v^*} \left[ \sup_{0 \leq t \leq T} \left\{ |D_t^{u^*,v^*} - D_t^{u,v^*}|^q \exp \left\{ q \int_0^t h_1(s, X_s^{0,x}, u_s, v_s^*) ds \right\} \right\} \right] \\ &\leq C \{ \mathbf{E}^{u,v^*} \left[ \sup_{0 \leq t \leq T} e^{pY_t^1} + \sup_{0 \leq t \leq T} e^{pY_t^{1,(u,v^*)}} \right] \\ &\quad + \mathbf{E}^{u,v^*} \left[ \sup_{0 \leq t \leq T} e^{q \cdot \frac{p}{p-q} C_h(1+|X_t^{0,x}|^\gamma)} \right] \}. \end{aligned} \quad (3.26)$$

Indeed, for any  $p > 1$ ,  $Y^1 \in \mathcal{D}_T^p(\mathbf{R}, d\mathbf{P}^{u,v^*})$ , since we assume  $Y_t^1 = \varpi^1(t, X_t^{0,x})$  where  $\varpi^1$  is deterministic and of subquadratic growth and finally (3.9). Meanwhile,  $Y^{1,(u,v^*)} \in \mathcal{D}_T^p(\mathbf{R}, d\mathbf{P}^{u,v^*})$  by (3.21). Therefore, (3.26) is finite. As the sequence  $(\Gamma_{\tau_n}^1)_{n \geq 1}$  converges to  $\Gamma_T^1 = 0$  as  $n \rightarrow \infty$ ,  $\mathbf{P}^{u,v^*}$ -a.s., it then also converges to 0 in  $\mathcal{L}^1(d\mathbf{P}^{u,v^*})$  since it is uniformly integral.

Next, by passing  $n$  to the limit on both sides of (3.25) and using the Fatou's lemma, we are able to show  $\mathbf{E}^{u,v^*} [\Gamma_t^1] = 0$ ,  $\forall t \leq T$ , which implies  $e^{Y_t^1} \leq e^{Y_t^{u,v^*}}$ ,  $\mathbf{P}$ -a.s., since the probabilities  $\mathbf{P}^{u,v^*}$  and  $\mathbf{P}$  are equivalent. Thus,  $e^{Y_0^1} = J^1(u^*, v^*) \leq e^{Y_0^{1,(u,v^*)}} = J^1(u, v^*)$ . In the same way, we can show that for arbitrary element  $v \in \mathcal{M}_2$ , then,  $e^{Y_0^2} = J^2(u^*, v^*) \leq e^{Y_0^{2,(u^*,v)}} = J^2(u^*, v)$ , which indicate that,  $(u^*, v^*)$  is an equilibrium point of the game.  $\square$

## 4 Existence of solutions for markovian BSDE

In Section 3, we provide the existence of the Nash equilibrium point under appropriate conditions. It remains to show that the BSDEs (3.20) have solutions as desired in Theorem 3.1. Therefore, in this section, we focus on this objective.

We firstly recall the notion of domination.

### 4.1 Measure domination

#### Definition 4.1. : $\mathcal{L}^q$ -Domination condition

Let  $q \in ]1, \infty[$  be fixed. For a given  $t_1 \in [0, T]$ , a family of probability measures  $\{\nu_1(s, dx), s \in [t_1, T]\}$  defined on  $\mathbf{R}^m$  is said to be  $\mathcal{L}^q$ -dominated by another family of probability measures  $\{\nu_0(s, dx), s \in [t_1, T]\}$ , if for any  $\delta \in (0, T - t_1]$ , there exists an application  $\phi_{t_1}^\delta : [t_1 + \delta, T] \times \mathbf{R}^m \rightarrow \mathbf{R}^+$  such that:

- (i)  $\nu_1(s, dx) ds = \phi_{t_1}^\delta(s, x) \nu_0(s, dx) ds$  on  $[t_1 + \delta, T] \times \mathbf{R}^m$ .
- (ii)  $\forall k \geq 1$ ,  $\phi_{t_1}^\delta(s, x) \in \mathcal{L}^q([t_1 + \delta, T] \times [-k, k]^m; \nu_0(s, dx) ds)$ .  $\square$



We then have:

**Lemma 4.1.** *Let  $q \in ]1, \infty[$  be fixed,  $(t_0, x_0) \in [0, T] \times \mathbf{R}^m$  and let  $(\theta_s^{t_0, x_0})_{t_0 \leq s \leq T}$  be the solution of SDE (3.1). If the diffusion coefficient function  $\sigma$  satisfies (2.1), then for any  $s \in (t_0, T]$ , the law  $\bar{\mu}(t_0, x_0; s, dx)$  of  $\theta_s^{t_0, x_0}$  has a density function  $\rho_{t_0, x_0}(s, x)$ , w.r.t. Lebesgue measure  $dx$ , which satisfies the following estimate:  $\forall (s, x) \in (t_0, T] \times \mathbf{R}^m$ ,*

$$\varrho_1(s-t_0)^{-\frac{m}{2}} \exp \left[ -\frac{\Lambda |x-x_0|^2}{s-t_0} \right] \leq \rho_{t_0, x_0}(s, x) \leq \varrho_2(s-t_0)^{-\frac{m}{2}} \exp \left[ -\frac{\lambda |x-x_0|^2}{s-t_0} \right] \quad (4.1)$$

where  $\varrho_1, \varrho_2, \Lambda, \lambda$  are real constants such that  $\varrho_1 \leq \varrho_2$  and  $\Lambda > \lambda$ . Moreover for any  $(t_1, x_1) \in [t_0, T] \times \mathbf{R}^m$ , the family of laws  $\{\bar{\mu}(t_1, x_1; s, dx), s \in [t_1, T]\}$  is  $L^q$ -dominated by  $\bar{\mu}(t_0, x_0; s, dx)$ .

*Proof.* Since  $\sigma$  satisfies (2.1) and  $b$  is bounded, then by Aronson's result (see [1]), the law  $\bar{\mu}(t_0, x_0; s, dx)$  of  $\theta_s^{t_0, x_0}$ ,  $s \in ]t_0, T]$ , has a density function  $\rho_{t_0, x_0}(s, x)$  which satisfies estimate (4.1).

Let us focus on the second claim of the lemma. Let  $(t_1, x_1) \in [t_0, T] \times \mathbf{R}^m$  and  $s \in (t_1, T]$ . Then

$$\rho_{t_1, x_1}(s, x) = [\rho_{t_1, x_1}(s, x) \rho_{t_0, x_0}^{-1}(s, x)] \rho_{t_0, x_0}(s, x) = \phi_{t_1}(s, x) \rho_{t_0, x_0}(s, x)$$

with

$$\phi_{t_1, x_1}(s, x) = [\rho_{t_1, x_1}(s, x) \rho_{t_0, x_0}^{-1}(s, x)], (s, x) \in (t_1, T] \times \mathbf{R}^m.$$

For any  $\delta \in (0, T-t_1]$ ,  $\phi_{t_1, x_1}$  is defined on  $[t_1 + \delta, T]$ . Moreover for any  $(s, x) \in [t_1 + \delta, T]$  it holds

$$\begin{aligned} \bar{\mu}(t_1, x_1; s, dx) ds &= \rho_{t_1, x_1}(s, x) dx ds \\ &= \phi_{t_1, x_1}(s, x) \rho_{t_0, x_0}(s, x) dx ds \\ &= \phi_{t_1, x_1}(s, x) \bar{\mu}(t_0, x_0; s, dx) ds. \end{aligned}$$

Next by (4.1), for any  $(s, x) \in [t_1 + \delta, T] \times \mathbf{R}^m$ ,

$$0 \leq \phi_{t_1, x_1}(s, x) \leq \frac{\varrho_2(s-t_1)^{-\frac{m}{2}}}{\varrho_1(s-t_0)^{-\frac{m}{2}}} \exp \left[ \frac{\Lambda |x-x_0|^2}{s-t_0} - \frac{\lambda |x-x_1|^2}{s-t_1} \right] \equiv \Phi_{t_1, x_1}(s, x).$$

It follows that for any  $k \geq 0$ , the function  $\Phi_{t_1, x_1}(s, x)$  is bounded on  $[t_1 + \delta, T] \times [-k, k]^m$  by a constant  $\kappa$  which depends on  $t_1, \delta, \Lambda, \lambda$  and  $k$ . Next let  $q \in (1, \infty)$ , then

$$\begin{aligned} \int_{t_1 + \delta}^T \int_{[-k, k]^m} \Phi(s, x)^q \bar{\mu}(t_0, x_0; s, dx) ds &\leq \kappa^q \int_{t_1 + \delta}^T \int_{[-k, k]^m} \bar{\mu}(t_0, x_0; s, dx) ds \\ &= \kappa^q \int_{t_1 + \delta}^T ds \mathbf{E}[1_{[-k, k]^m}(\theta_s^{t_0, x_0})] \leq \kappa^q T. \end{aligned}$$

Thus  $\Phi$  and then  $\phi$  belong to  $\mathcal{L}^q([t_1 + \delta, T] \times [-k, k]^m; \nu_0(s, dx) ds)$ . It follows that the family of measures  $\{\bar{\mu}(t_1, x_1; s, dx), s \in [t_1, T]\}$  is  $\mathcal{L}^q$ -dominated by  $\bar{\mu}(t_0, x_0; s, dx)$ .  $\square$

As a by-product we have:

**Corollary 4.1.** *Let  $x \in \mathbf{R}^m$  be fixed,  $t \in [0, T]$ ,  $s \in (t, T]$  and  $\mu(t, x; s, dy)$  the law of  $X_s^{t, x}$ , i.e.,*

$$\forall A \in \mathcal{B}(\mathbf{R}^m), \mu(t, x; s, A) = \mathbf{P}(X_s^{t, x} \in A).$$

*If  $\sigma$  satisfies (2.1), then for any  $q \in (1, \infty)$ , the family of laws  $\{\mu(t, x; s, dy), s \in [t, T]\}$  is  $\mathcal{L}^q$ -dominated by  $\{\mu(0, x; s, dy), s \in [t, T]\}$ .  $\square$*

## 4.2 Existence of solutions for BSDE (3.20)

Now, we are well-prepared to provide the existence of solution for BSDE (3.20).

**Theorem 4.1.** *Let  $x \in \mathbf{R}^m$  be fixed. Then under Assumptions (A1)-(A3), there exist two pairs of  $\mathcal{P}$ -measurable processes  $(Y^i, Z^i)$  with values in  $\mathbf{R}^{1+m}$ ,  $i = 1, 2$ , and two deterministic functions  $\varpi^i(t, x)$  which are of subquadratic growth, i.e.  $|\varpi^i(t, x)| \leq C(1 + |x|^\gamma)$  with  $1 < \gamma < 2$ ,  $i = 1, 2$  such that,*

$$\begin{cases} \mathbf{P}\text{-a.s.}, \forall t \leq T, Y_t^i = \varpi^i(t, X_t^{0,x}) \text{ and } Z^i \text{ is } dt\text{-square integrable } \mathbf{P}\text{-a.s.}; \\ Y_t^i = g^i(X_t^{0,x}) + \int_t^T \{H_i(s, X_s^{0,x}, Z_s^i, (u^*, v^*)(s, X_s^{0,x}, Z_s^1, Z_s^2)) + \frac{1}{2} |Z_s^i|^2\} ds \\ \quad - \int_t^T Z_s^i dB_s, \quad \forall t \leq T. \end{cases} \quad (4.2)$$

*Proof.* We shall divide the proof into several steps. Our plan is the following. We apply the exponential exchange (see e.g. [17]) to eliminate the quadratic term in the generator. The pair of the solution processes (resp. the generator) is denoted by  $(\bar{Y}, \bar{Z})$  (resp.  $G$ ). We then approximate the new generator  $G$  by the Lipschitz continuous ones, which we denoted by  $G^n$ , such that the classical results about BSDE can be applied. It follows that, for each  $n$ , the BSDE with generator  $G$  being replaced by  $G^n$ , has a solution  $(\bar{Y}^n, \bar{Z}^n)$ . After that, we give the uniform estimates of the solutions, as well as the convergence property. In the convergence step, the measure domination property Corollary 4.1 plays a crucial role in passing from the weak limit to a strong sense one. Finally, we verify that the limits of the sequences are exactly the solutions of the BSDE.

**Step 1.** *Exponential exchange and approximation.*

For  $t \in [0, T]$ , and  $i = 1, 2$ , let us denote by:

$$\begin{cases} \bar{Y}_t^i = e^{Y_t^i}; \\ \bar{Z}_t^i = \bar{Y}_t^i Z_t^i. \end{cases} \quad (4.3)$$

Then, BSDE (4.2) reads, for  $t \in [0, T]$  and  $i = 1, 2$ ,

$$\begin{aligned} \bar{Y}_t^i &= e^{g^i(X_t^{0,x})} + \int_t^T \mathbb{1}_{\bar{Y}_s^i > 0} \{ \bar{Z}_s^i \sigma^{-1}(s, X_s^{0,x}) f(s, X_s^{0,x}, (u^*, v^*)(s, X_s^{0,x}, \frac{\bar{Z}_s^1}{\bar{Y}_s^1}, \frac{\bar{Z}_s^2}{\bar{Y}_s^2})) \\ &\quad + \bar{Y}_s^i h_i(s, X_s^{0,x}, (u^*, v^*)(s, X_s^{0,x}, \frac{\bar{Z}_s^1}{\bar{Y}_s^1}, \frac{\bar{Z}_s^2}{\bar{Y}_s^2})) \} ds - \int_t^T \bar{Z}_s^i dB_s. \end{aligned} \quad (4.4)$$

Let us deal with the case  $i = 1$  for example and the case  $i = 2$  follows in the same way. Inspiring by the mollify technique in [13], we first denote here the generator of (4.4) by  $G^1 : [0, T] \times \mathbf{R}^m \times \mathbf{R}^{+*} \times \mathbf{R}^{+*} \times \mathbf{R}^{2m} \rightarrow \mathbf{R}$  (by  $\mathbf{R}^{+*}$ , we refer to  $\mathbf{R}^+ \setminus \{0\}$ ), i.e.

$$\begin{aligned} G^1(s, x, y^1, y^2, z^1, z^2) &= \mathbb{1}_{y^1 > 0} \{ z^1 \sigma^{-1}(s, x) f(s, x, (u^*, v^*)(s, x, \frac{z^1}{y^1}, \frac{z^2}{y^2})) \\ &\quad + y^1 h(s, x, (u^*, v^*)(s, x, \frac{z^1}{y^1}, \frac{z^2}{y^2})) \} \end{aligned}$$

which is still continuous w.r.t  $(y^1, y^2, z^1, z^2)$  considering the Assumption (A3)-(ii) and the transformation (4.3). Let  $\xi$  be an element of  $C^\infty(\mathbf{R}^{+*} \times \mathbf{R}^{+*} \times \mathbf{R}^{2m}, \mathbf{R})$  with compact support and satisfying:

$$\int_{\mathbf{R}^{+*} \times \mathbf{R}^{+*} \times \mathbf{R}^{2m}} \xi(y^1, y^2, z^1, z^2) dy^1 dy^2 dz^1 dz^2 = 1.$$

For  $(t, x, y^1, y^2, z^1, z^2) \in [0, T] \times \mathbf{R}^m \times \mathbf{R}^{+*} \times \mathbf{R}^{+*} \times \mathbf{R}^{2m}$ , we set,

$$\begin{aligned} \tilde{G}^{1n}(t, x, y^1, y^2, z^1, z^2) &= \int_{\mathbf{R}^{+*} \times \mathbf{R}^{+*} \times \mathbf{R}^{2m}} n^4 G^1(s, \varphi_n(x), p^1, p^2, q^1, q^2) \\ &\quad \xi(n(y^1 - p^1), n(y^2 - p^2), n(z^1 - q^1), n(z^2 - q^2)) dp^1 dp^2 dq^1 dq^2, \end{aligned}$$

where  $\varphi_n(x) = ((x_j \vee (-n)) \wedge n)_{j=1,2,\dots,m}$ , for  $x = (x_j)_{j=1,2,\dots,m} \in \mathbf{R}^m$ . We next define  $\psi \in C^\infty(\mathbf{R}^{2+2m}, \mathbf{R})$  by,

$$\psi(y^1, y^2, z^1, z^2) = \begin{cases} 1, & |y^1|^2 + |y^2|^2 + |z^1|^2 + |z^2|^2 \leq 1, \\ 0, & |y^1|^2 + |y^2|^2 + |z^1|^2 + |z^2|^2 \geq 4. \end{cases}$$

Then, we define the measurable function sequence  $(G^{1n})_{n \geq 1}$  as follows:  $\forall (t, x, y^1, y^2, z^1, z^2) \in [0, T] \times \mathbf{R}^m \times \mathbf{R} \times \mathbf{R} \times \mathbf{R}^{2m}$ ,

$$G^{1n}(t, x, y^1, y^2, z^1, z^2) = \psi\left(\frac{y^1}{n}, \frac{y^2}{n}, \frac{z^1}{n}, \frac{z^2}{n}\right) \tilde{G}^{1n}(t, x, \psi_n(y^1), \psi_n(y^2), z^1, z^2),$$

where for each  $n$ ,  $\psi_n(y)$  is a continuous function for  $y \in \mathbf{R}$ , and  $\psi_n(y) = 1/n$  if  $y \leq 0$ ;  $\psi_n(y) = y$  if  $y \geq 1/n$ . We have the following properties:

$$\left\{ \begin{array}{l} (a) \ G^{1n} \text{ is uniformly lipschitz w.r.t } (y^1, y^2, z^1, z^2); \\ (b) \ |G^{1n}(t, x, y^1, y^2, z^1, z^2)| \leq C_f C_\sigma (1 + |\varphi_n(x)|) |z^1| + C_h (1 + |\varphi_n(x)|^\gamma) (y^1)^+; \\ (c) \ |G^{1n}(t, x, y^1, y^2, z^1, z^2)| \leq c_n, \text{ for any } (t, x, y^1, y^2, z^1, z^2); \\ (d) \text{ For any } (t, x) \in [0, T] \times \mathbf{R}^m, \varepsilon > 0 \text{ and } \mathbf{K} \text{ a compact subset of } [\varepsilon, \frac{1}{\varepsilon}]^2 \times \mathbf{R}^{2m}, \\ \quad \sup_{(y^1, y^2, z^1, z^2) \in \mathbf{K}} |G^{1n}(t, x, y^1, y^2, z^1, z^2) - G^1(t, x, y^1, y^2, z^1, z^2)| \rightarrow 0, \text{ as } n \rightarrow \infty. \end{array} \right. \quad (4.5)$$

The same technique provides the sequence  $(G^{2n})_{n \geq 1}$ , which is indeed, the approximation of function  $G^2$ . For each  $n \geq 1$  and  $(t, x) \in [0, T] \times \mathbf{R}^m$ , it is a direct result of (4.5)-(a) that (see [18]), there exist two pairs of processes  $(\bar{Y}_s^{1n;(t,x)}, \bar{Z}_s^{1n;(t,x)})_{t \leq s \leq T}, (\bar{Y}_s^{2n;(t,x)}, \bar{Z}_s^{2n;(t,x)})_{t \leq s \leq T} \in \mathcal{S}_{t,T}^2(\mathbf{R}) \times \mathcal{H}_{t,T}^2(\mathbf{R}^m)$ , which sat-

isfy, for  $s \in [t, T]$ ,

$$\begin{cases} \bar{Y}_s^{1n;(t,x)} = e^{g^1(X_T^{t,x})} + \int_s^T G^{1n}(r, X_r^{t,x}, \bar{Y}_r^{1n;(t,x)}, \bar{Y}_r^{2n;(t,x)}, \bar{Z}_r^{1n;(t,x)}, \bar{Z}_r^{2n;(t,x)})dr \\ \quad - \int_s^T \bar{Z}_r^{1n;(t,x)} dB_r; \\ \bar{Y}_s^{2n;(t,x)} = e^{g^2(X_T^{t,x})} + \int_s^T G^{2n}(r, X_r^{t,x}, \bar{Y}_r^{1n;(t,x)}, \bar{Y}_r^{2n;(t,x)}, \bar{Z}_r^{1n;(t,x)}, \bar{Z}_r^{2n;(t,x)})dr \\ \quad - \int_s^T \bar{Z}_r^{2n;(t,x)} dB_r. \end{cases} \quad (4.6)$$

Meanwhile, the properties (4.5)-(a),(c) and the result of El karoui et al. (ref. [6]) yield that, there exist two sequences of deterministic measurable applications  $\varsigma^{1n}(\text{resp. } \varsigma^{2n}) : [0, T] \times \mathbf{R}^m \rightarrow \mathbf{R}$  and  $\mathfrak{z}^{1n}(\text{resp. } \mathfrak{z}^{2n}) : [0, T] \times \mathbf{R}^m \rightarrow \mathbf{R}^m$  such that for any  $s \in [t, T]$ ,

$$\bar{Y}_s^{1n;(t,x)} = \varsigma^{1n}(s, X_s^{t,x}) \quad (\text{resp. } \bar{Y}_s^{2n;(t,x)} = \varsigma^{2n}(s, X_s^{t,x})) \quad (4.7)$$

and

$$\bar{Z}_s^{1n;(t,x)} = \mathfrak{z}^{1n}(s, X_s^{t,x}) \quad (\text{resp. } \bar{Z}_s^{2n;(t,x)} = \mathfrak{z}^{2n}(s, X_s^{t,x})).$$

Besides, we have the following deterministic expression: for  $i = 1, 2$ , and  $n \geq 1$ ,

$$\varsigma^{in}(t, x) = \mathbf{E} \left[ e^{g^i(X_T^{t,x})} + \int_t^T F^{in}(s, X_s^{t,x}) ds \right], \quad \forall (t, x) \in [0, T] \times \mathbf{R}^m, \quad (4.8)$$

where,

$$F^{in}(s, x) = G^{in}(s, x, \varsigma^{1n}(s, x), \varsigma^{2n}(s, x), \mathfrak{z}^{1n}(s, x), \mathfrak{z}^{2n}(s, x)).$$

**Step 2.** *Uniform integrability of  $(\bar{Y}^{1n;(t,x)})_{n \geq 1}$  for fixed  $(t, x) \in [0, T] \times \mathbf{R}^m$ .*

In this step, we will deal with the case of  $i = 1$ , the case of  $i = 2$  can be treated in a similar way. For each  $n \geq 1$ , let us consider BSDE as follows, for  $s \in [t, T]$ ,

$$\begin{aligned} \tilde{Y}_s^{1n} &= e^{g^1(X_T^{t,x})} + \int_s^T \left\{ C_f C_\sigma (1 + |\varphi_n(X_r^{t,x})|) |\tilde{Z}_r^{1n}| + C_h (1 + |\varphi_n(X_r^{t,x})|^\gamma) (\tilde{Y}_r^{1n})^+ \right\} dr \\ &\quad - \int_s^T \tilde{Z}_r^{1n} dB_r. \end{aligned} \quad (4.9)$$

Obviously, for any  $x \in \mathbf{R}^m$  and integer  $n \geq 1$ , the application which to  $(y, z) \in \mathbf{R}^{1+m}$  associates  $C_f C_\sigma (1 + \varphi_n(x)) |z| + C_h (1 + \varphi_n(x))^\gamma y^+$  is Lipchitz continuous. Besides,  $e^{g^1(X_T^{t,x})} \in \mathcal{L}^p(d\mathbf{P})$ ,  $\forall p \geq 1$  which is the consequence of Assumption (A2)-(iii) and (3.7). Therefore, from the result of Pardoux and Peng [19], we know that a pair of solutions  $(\tilde{Y}_s^{1n}, \tilde{Z}_s^{1n})_{t \leq s \leq T} \in \mathcal{S}_{t,T}^p(\mathbf{R}) \times \mathcal{H}_{t,T}^p(\mathbf{R}^m)$  exists for any  $p > 1$ . Moreover through an adaptation of the result given by El Karoui et al. (1997,[6]), we can infer the existence of deterministic measurable function  $\tilde{\varsigma}^{1n} : [0, T] \times \mathbf{R}^m \rightarrow \mathbf{R}$  such that, for any  $s \in [t, T]$ ,

$$\tilde{Y}_s^{1n} = \tilde{\varsigma}^{1n}(s, X_s^{t,x}). \quad (4.10)$$

Next let us consider the process

$$B_s^n = B_s - \int_0^s 1_{[t,T]}(r) C_f C_\sigma (1 + |\varphi_n(X_r^{t,x})|) \text{sign}(\tilde{Z}_r^{1n}) dr, \quad 0 \leq s \leq T,$$

which is, thanks to Girsanov's Theorem, a Brownian motion under the probability  $\mathbf{P}^n$  on  $(\Omega, \mathcal{F})$  whose density with respect to  $\mathbf{P}$  is

$$\zeta_T := \zeta_T \{C_f C_\sigma (1 + |\varphi_n(X_s^{t,x})|) \text{sign}(\tilde{Z}_s^{1n}) 1_{[t,T]}(s)\},$$

where for any  $z = (z^i)_{i=1,\dots,d} \in \mathbf{R}^m$ ,  $\text{sign}(z) = (1_{[|z^i| \neq 0]} \frac{z^i}{|z^i|})_{i=1,\dots,d}$  and  $\zeta_T(\cdot)$  is defined by (2.4). Then (4.9) becomes

$$\tilde{Y}_s^{1n} = e^{g^1(X_T^{t,x})} + \int_s^T C_h (1 + |\varphi_n(X_r^{t,x})|^\gamma) (\tilde{Y}_r^{1n})^+ dr - \int_s^T \tilde{Z}_r^{1n} dB_r^n, \quad t \leq s \leq T.$$

Therefore, taking into account of (4.10), we deduce,

$$\tilde{\zeta}^{1n}(t, x) = \mathbf{E}^n \left[ e^{g^1(X_T^{t,x}) + \int_t^T C_h (1 + |\varphi_n(X_s^{t,x})|^\gamma) ds} \middle| \mathcal{F}_t \right],$$

where  $\mathbf{E}^n$  is the expectation under probability  $\mathbf{P}^n$ . Taking the expectation on both sides under the probability  $\mathbf{P}^n$  and considering  $\tilde{\zeta}^{1n}(t, x)$  is deterministic, one obtains,

$$\tilde{\zeta}^{1n}(t, x) = \mathbf{E}^n \left[ e^{g^1(X_T^{t,x}) + \int_t^T C_h (1 + |\varphi_n(X_s^{t,x})|^\gamma) ds} \right].$$

Then by the Assumption (A2)-(iii) we have:  $\forall (t, x) \in [0, T] \times \mathbf{R}^m$ ,

$$\begin{aligned} |\tilde{\zeta}^{1n}(t, x)| &\leq \mathbf{E}^n \left[ e^{C \sup_{0 \leq s \leq T} (1 + |X_s^{t,x}|^\gamma)} \right] \\ &= \mathbf{E} \left[ e^{C \sup_{0 \leq s \leq T} (1 + |X_s^{t,x}|^\gamma)} \cdot \zeta_T \right]. \end{aligned}$$

By Lemma 3.2, there exists some  $1 < p_0 < 2$  (which does not depend on  $(t, x)$ ), such that  $\mathbf{E}[|\zeta_T|^{p_0}] < \infty$ . Applying Young's inequality, besides, considering (3.7) yield that,

$$\begin{aligned} |\tilde{\zeta}^{1n}(t, x)| &\leq \mathbf{E} \left[ e^{\frac{C p_0}{p_0 - 1} \sup_{0 \leq s \leq T} (1 + |X_s^{t,x}|^\gamma)} \right] + \mathbf{E}[|\zeta_T|^{p_0}] \\ &\leq e^{C(1+|x|^\gamma)}. \end{aligned}$$

Next taking into account point (4.5)-(b) and using comparison Theorem of BSDEs, we obtain for any  $s \in [t, T]$ ,

$$\tilde{Y}_s^{1n} = \tilde{\zeta}^{1n}(s, X_s^{t,x}) \geq \bar{Y}_s^{1n;(t,x)} = \varsigma^{1n}(s, X_s^{t,x}).$$

Then, by choosing  $s = t$ , we get that  $\varsigma^{1n}(t, x) \leq e^{C(1+|x|^\gamma)}$ ,  $(t, x) \in [0, T] \times \mathbf{R}^m$ . But in a similar way one can show that for any  $(t, x) \in [0, T] \times \mathbf{R}^m$ ,  $\varsigma^{1n}(t, x) \geq e^{-C(1+|x|^\gamma)}$ . Therefore,

$$e^{-C(1+|x|^\gamma)} \leq \varsigma^{1n}(t, x) \leq e^{C(1+|x|^\gamma)}, \quad (t, x) \in [0, T] \times \mathbf{R}^m. \quad (4.11)$$

By (4.11), (4.7) and (3.7), we conclude,  $\bar{Y}_s^{1n;(t,x)} \in \mathcal{S}_{t,T}^p(\mathbf{R}^m)$  holds, i.e., for any  $p > 1$ , we have,

$$\mathbf{E} \left[ \sup_{t \leq s \leq T} |\bar{Y}_s^{1n;(t,x)}|^p \right] < \infty. \quad (4.12)$$

**Step 3.** *Uniform integrability of  $(\bar{Z}_s^{1n;(t,x)})_{t \leq s \leq T}$ .*

Recalling the equation (4.6) and making use of Itô's formula with  $(\bar{Y}_s^{1n;(t,x)})^2$ , we obtain, in a standard way, the following result.

There exists a constant  $C$  independent of  $n$  and  $t, x$  such that for any  $t \leq T$ , for  $i = 1, 2$ ,

$$\mathbf{E} \left[ \int_t^T |\bar{Z}_s^{1n;(t,x)}|^2 ds \right] \leq C. \quad (4.13)$$

The proof is omitted for conciseness.

**Step 4.** *There exists a subsequence of  $((\bar{Y}_s^{1n;(0,x)}, \bar{Z}_s^{1n;(0,x)})_{0 \leq s \leq T})_{n \geq 1}$  which converges respectively to  $(\bar{Y}_s^1, \bar{Z}_s^1)_{0 \leq s \leq T}$ , solution of the BSDE (4.4). Moreover,  $\bar{Y}_s^1 > 0$ ,  $\forall s \in [0, T]$ ,  $\mathbf{P}$ -a.s.*

Let us recall the expression (4.8) for case  $i = 1$ ,

$$\varsigma^{1n}(t, x) = \mathbf{E} \left[ e^{g^1(X_T^{t,x})} + \int_t^T F^{1n}(s, X_s^{t,x}) ds \right], \quad \forall (t, x) \in [0, T] \times \mathbf{R}^m. \quad (4.14)$$

We now apply property (4.5)-(b) in Step 1 combined with the uniform estimates (4.12), (4.13) and the Young's inequality to show that, for  $1 < q < 2$ ,

$$\begin{aligned} & \mathbf{E} \left[ \int_0^T |F^{1n}(s, X_s^{0,x})|^q ds \right] \\ & \leq C \mathbf{E} \left[ \int_0^T (1 + |\varphi_n(X_s^{0,x})|)^q |\bar{Z}_s^{1n;(0,x)}|^q + (1 + |\varphi_n(X_s^{0,x})|)^{\gamma q} |\bar{Y}_s^{1n;(0,x)}|^q ds \right] \\ & \leq C \mathbf{E} \left[ \left( \int_0^T |\bar{Z}_s^{1n;(0,x)}|^2 ds \right)^{\frac{q}{2}} \left( \int_0^T (1 + |X_s^{0,x}|)^{\frac{2q}{2-q}} ds \right)^{\frac{2-q}{2}} \right] \\ & \quad + C \mathbf{E} \left[ \sup_{0 \leq s \leq T} |\bar{Y}_s^{1n;(0,x)}|^q \cdot \int_0^T (1 + |X_s^{0,x}|)^{\gamma q} ds \right] \\ & \leq C \{ \mathbf{E} \left[ \int_0^T |\bar{Z}_s^{1n;(0,x)}|^2 ds \right] + \mathbf{E} \left[ \sup_{0 \leq s \leq T} |\bar{Y}_s^{1n;(0,x)}|^2 \right] + 1 \} \\ & < \infty. \end{aligned} \quad (4.15)$$

Therefore, there exists a sub-sequence  $\{n_k\}$  (for notation simplification, we still denote it by  $\{n\}$ ) and a  $\mathcal{B}([0, T]) \otimes \mathcal{B}(\mathbf{R}^m)$ -measurable deterministic function  $F^1(s, y)$  such that:

$$F^{1n} \rightarrow F^1 \text{ weakly in } \mathcal{L}^q([0, T] \times \mathbf{R}^m; \mu(0, x; s, dy) ds). \quad (4.16)$$

Next we aim to prove that  $(\varsigma^{1n}(t, x))_{n \geq 1}$  is a Cauchy sequence for each  $(t, x) \in [0, T] \times$

$\mathbf{R}^m$ . Now let  $(t, x)$  be fixed,  $\eta > 0$ ,  $k, n$  and  $m \geq 1$  be integers. From (4.14), we have,

$$\begin{aligned} |\varsigma^{1n}(t, x) - \varsigma^{1m}(t, x)| &= \left| \mathbf{E} \left[ \int_t^T F^{1n}(s, X_s^{t,x}) - F^{1m}(s, X_s^{t,x}) ds \right] \right| \\ &\leq \mathbf{E} \left[ \int_t^{t+\eta} |F^{1n}(s, X_s^{t,x}) - F^{1m}(s, X_s^{t,x})| ds \right] \\ &\quad + \left| \mathbf{E} \left[ \int_{t+\eta}^T (F^{1n}(s, X_s^{t,x}) - F^{1m}(s, X_s^{t,x})) \cdot \mathbb{1}_{\{|X_s^{t,x}| \leq k\}} ds \right] \right| \\ &\quad + \left| \mathbf{E} \left[ \int_{t+\eta}^T (F^{1n}(s, X_s^{t,x}) - F^{1m}(s, X_s^{t,x})) \cdot \mathbb{1}_{\{|X_s^{t,x}| > k\}} ds \right] \right|, \end{aligned}$$

where on the right side, noticing (4.15), we obtain,

$$\begin{aligned} &\mathbf{E} \left[ \int_t^{t+\eta} |F^{1n}(s, X_s^{t,x}) - F^{1m}(s, X_s^{t,x})| ds \right] \\ &\leq \eta^{\frac{q-1}{q}} \{ \mathbf{E} \left[ \int_0^T |F^{1n}(s, X_s^{t,x}) - F^{1m}(s, X_s^{t,x})|^q ds \right] \}^{\frac{1}{q}} \leq C \eta^{\frac{q-1}{q}}. \end{aligned}$$

At the same time, Corollary 4.1 associates with the  $\mathcal{L}^{\frac{q}{q-1}}$ -domination property implies:

$$\begin{aligned} &\left| \mathbf{E} \left[ \int_{t+\eta}^T (F^{1n}(s, X_s^{t,x}) - F^{1m}(s, X_s^{t,x})) \cdot \mathbb{1}_{\{|X_s^{t,x}| \leq k\}} ds \right] \right| \\ &= \left| \int_{\mathbf{R}^m} \int_{t+\eta}^T (F^{1n}(s, \eta) - F^{1m}(s, \eta)) \cdot \mathbb{1}_{\{|\eta| \leq k\}} \mu(t, x; s, d\eta) ds \right| \\ &= \left| \int_{\mathbf{R}^m} \int_{t+\eta}^T (F^{1n}(s, \eta) - F^{1m}(s, \eta)) \cdot \mathbb{1}_{\{|\eta| \leq k\}} \phi_{t,x}(s, \eta) \mu(0, x; s, d\eta) ds \right|. \end{aligned}$$

Since  $\phi_{t,x}(s, \eta) \in \mathcal{L}^{\frac{q}{q-1}}([t+\eta, T] \times [-k, k]^m; \mu(0, x; s, d\eta) ds)$ , for  $k \geq 1$ , it follows from (4.16) that for each  $(t, x) \in [0, T] \times \mathbf{R}^m$ , we have,

$$\mathbf{E} \left[ \int_{t+\eta}^T (F^{1n}(s, X_s^{t,x}) - F^{1m}(s, X_s^{t,x})) \mathbb{1}_{\{|X_s^{t,x}| \leq k\}} ds \right] \rightarrow 0 \text{ as } n, m \rightarrow \infty.$$

Finally,

$$\begin{aligned} &\left| \mathbf{E} \left[ \int_{t+\eta}^T (F^{1n}(s, X_s^{t,x}) - F^{1m}(s, X_s^{t,x})) \cdot \mathbb{1}_{\{|X_s^{t,x}| > k\}} ds \right] \right| \\ &\leq C \{ \mathbf{E} \left[ \int_{t+\eta}^T \mathbb{1}_{\{|X_s^{t,x}| > k\}} ds \right] \}^{\frac{q-1}{q}} \{ \mathbf{E} \left[ \int_{t+\eta}^T |F^{1n}(s, X_s^{t,x}) - F^{1m}(s, X_s^{t,x})|^q ds \right] \}^{\frac{1}{q}} \\ &\leq C k^{-\frac{q-1}{q}} \end{aligned}$$

Therefore, for each  $(t, x) \in [0, T] \times \mathbf{R}^m$ ,  $(\varsigma^{1n}(t, x))_{n \geq 1}$  is a Cauchy sequence and then there exists a borelian application  $\varsigma^1$  on  $[0, T] \times \mathbf{R}^m$ , such that for each

$(t, x) \in [0, T] \times \mathbf{R}^m$ ,  $\lim_{n \rightarrow \infty} \varsigma^{1n}(t, x) = \varsigma^1(t, x)$ , which indicates that for  $t \in [0, T]$ ,  $\lim_{n \rightarrow \infty} \bar{Y}_t^{1n; (0, x)}(\omega) = \varsigma^1(t, X_t^{0, x})$ ,  $\mathbf{P}$ -a.s. Taking account of (4.12) and the Lebesgue dominated convergence theorem, we obtain the sequence  $((\bar{Y}_t^{1n; (0, x)})_{0 \leq t \leq T})_{n \geq 1}$  converges to  $\bar{Y}^1 = (\varsigma^1(t, X_t^{0, x}))_{0 \leq t \leq T}$  in  $\mathcal{L}^p([0, T] \times \mathbf{R}^m)$  for any  $p > 1$ , that is:

$$\mathbf{E} \left[ \int_0^T |\bar{Y}_t^{1n; (0, x)} - \bar{Y}_t^1|^p dt \right] \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (4.17)$$

Next, we will show that for any  $p > 1$ ,  $\bar{Z}^{1n; (0, x)} = (\bar{Z}_t^{1n; (0, x)})_{0 \leq t \leq T}$  has a limit in  $\mathcal{H}_T^2(\mathbf{R}^m)$ . Besides,  $(\bar{Y}^{1n; (0, x)})_{n \geq 1}$  is convergent in  $\mathcal{S}_T^2(\mathbf{R})$  as well.

We now focus on the first claim. For  $n, m \geq 1$  and  $0 \leq t \leq T$ , using Itô's formula with  $(\bar{Y}_t^{1n} - \bar{Y}_t^{1m})^2$  (we omit the subscript  $(0, x)$  for convenience) and considering (4.5)-(b) in Step 1, we get,

$$\begin{aligned} & |\bar{Y}_t^{1n} - \bar{Y}_t^{1m}|^2 + \int_t^T |\bar{Z}_s^{1n} - \bar{Z}_s^{1m}|^2 ds \\ &= 2 \int_t^T (\bar{Y}_s^{1n} - \bar{Y}_s^{1m}) (G^{1n}(s, X_s^{0, x}, \bar{Y}_s^{1n}, \bar{Y}_s^{2n}, \bar{Z}_s^{1n}, \bar{Z}_s^{2n}) - \\ & \quad - G^{1m}(s, X_s^{0, x}, \bar{Y}_s^{1m}, \bar{Y}_s^{2m}, \bar{Z}_s^{1m}, \bar{Z}_s^{2m})) ds - 2 \int_t^T (\bar{Y}_s^{1n} - \bar{Y}_s^{1m}) (\bar{Z}_s^{1n} - \bar{Z}_s^{1m}) dB_s \\ &\leq C \int_t^T |\bar{Y}_s^{1n} - \bar{Y}_s^{1m}| \left[ (|\bar{Z}_s^{1n}| + |\bar{Z}_s^{1m}|)(1 + |X_s^{0, x}|) + (|\bar{Y}_s^{1n}| + |\bar{Y}_s^{1m}|)(1 + |X_s^{0, x}|)^\gamma \right] ds \\ & \quad - 2 \int_t^T (\bar{Y}_s^{1n} - \bar{Y}_s^{1m}) (\bar{Z}_s^{1n} - \bar{Z}_s^{1m}) dB_s. \end{aligned}$$

Since for any  $x, y, z \in \mathbf{R}$ ,  $|xyz| \leq \frac{1}{a}|x|^a + \frac{1}{b}|y|^b + \frac{1}{c}|z|^c$  with  $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$ , then, for any  $\varepsilon > 0$ , we have,

$$\begin{aligned} & |\bar{Y}_t^{1n} - \bar{Y}_t^{1m}|^2 + \int_t^T |\bar{Z}_s^{1n} - \bar{Z}_s^{1m}|^2 ds \\ &\leq C \left\{ \frac{\varepsilon^2}{2} \int_t^T (|\bar{Z}_s^{1n}| + |\bar{Z}_s^{1m}|)^2 ds + \frac{\varepsilon^4}{4} \int_t^T (1 + |X_s^{0, x}|)^4 ds + \frac{1}{4\varepsilon^8} \int_t^T |\bar{Y}_s^{1n} - \bar{Y}_s^{1m}|^4 ds \right. \\ & \quad \left. + \frac{\varepsilon^2}{2} \int_t^T (|\bar{Y}_s^{1n}| + |\bar{Y}_s^{1m}|)^2 ds + \frac{\varepsilon^4}{4} \int_t^T (1 + |X_s^{0, x}|)^{4\gamma} ds + \frac{1}{4\varepsilon^8} \int_t^T |\bar{Y}_s^{1n} - \bar{Y}_s^{1m}|^4 ds \right\} \\ & \quad - 2 \int_t^T (\bar{Y}_s^{1n} - \bar{Y}_s^{1m}) (\bar{Z}_s^{1n} - \bar{Z}_s^{1m}) dB_s. \end{aligned} \quad (4.18)$$

Taking now  $t = 0$  in (4.18), expectation on both sides and the limit w.r.t.  $n$  and  $m$ , we deduce that,

$$\limsup_{n, m \rightarrow \infty} \mathbf{E} \left[ \int_0^T |\bar{Z}_s^{1n} - \bar{Z}_s^{1m}|^2 ds \right] \leq C \left\{ \frac{\varepsilon^2}{2} + \frac{\varepsilon^4}{4} \right\}, \quad (4.19)$$

due to (4.13), (3.6) and the convergence of (4.17). As  $\varepsilon$  is arbitrary, then the sequence  $(\bar{Z}^{1n})_{n \geq 1}$  is convergent in  $\mathcal{H}_T^2(\mathbf{R}^m)$  to a process  $Z^1$ .



Now, returning to inequality (4.18), taking the supremum over  $[0, T]$  and using BDG's inequality, we obtain that,

$$\begin{aligned} & \mathbf{E} \left[ \sup_{0 \leq t \leq T} |\bar{Y}_t^{1n} - \bar{Y}_t^{1m}|^2 + \int_0^T |\bar{Z}_s^{1n} - \bar{Z}_s^{1m}|^2 ds \right] \\ & \leq C \left\{ \frac{\varepsilon^2}{2} + \frac{\varepsilon^4}{4} \right\} + \frac{1}{4} \mathbf{E} \left[ \sup_{0 \leq t \leq T} |\bar{Y}_t^{1n} - \bar{Y}_t^{1m}|^2 \right] + 4 \mathbf{E} \left[ \int_0^T |\bar{Z}_s^{1n} - \bar{Z}_s^{1m}|^2 ds \right] \end{aligned}$$

which implies that

$$\limsup_{n, m \rightarrow \infty} \mathbf{E} \left[ \sup_{0 \leq t \leq T} |\bar{Y}_t^{1n} - \bar{Y}_t^{1m}|^2 \right] = 0,$$

since  $\varepsilon$  is arbitrary and (4.19). Thus, the sequence of  $(\bar{Y}^{1n})_{n \geq 1}$  converges to  $\bar{Y}^1$  in  $\mathcal{S}_T^2(\mathbf{R})$  which is a continuous process.

Next, note that since  $\varsigma^{1n}(s, x) \geq e^{-C(1+|x|^\gamma)}$ , then,  $\bar{Y}_s^1 > 0$ ,  $\forall s \leq T$ ,  $\mathbf{P}$ -a.s.

Finally, repeat the procedure for player  $i = 2$ , we have also the convergence of  $(\bar{Z}^{2n})_{n \geq 1}$  (resp.  $(\bar{Y}^{2n})_{n \geq 1}$ ) in  $\mathcal{H}_T^2(\mathbf{R}^m)$  (resp.  $\mathcal{S}_T^2(\mathbf{R})$ ) to  $\bar{Z}^2$  (resp.  $\bar{Y}^2 = \varsigma^2(\cdot, X^{0,x})$ ).

**Step 5.** The limit process  $(\bar{Y}_t^i, \bar{Z}_t^i)_{0 \leq t \leq T}$  ( $i=1,2$ ) is the solution of BSDE (4.4).

Indeed, we need to show that (for case  $i = 1$ ):

$$F^1(t, X_t^{0,x}) = G^1(t, X_t^{0,x}, \bar{Y}_t^1, \bar{Y}_t^2, \bar{Z}_t^1, \bar{Z}_t^2) \quad dt \otimes d\mathbf{P} - a.s.$$

For  $k \geq 1$ , we have,

$$\begin{aligned} & \mathbf{E} \left[ \int_0^T |G^{1n}(s, X_s^{0,x}, \bar{Y}_s^{1n}, \bar{Y}_s^{2n}, \bar{Z}_s^{1n}, \bar{Z}_s^{2n}) - G^1(s, X_s^{0,x}, \bar{Y}_s^1, \bar{Y}_s^2, \bar{Z}_s^1, \bar{Z}_s^2)| ds \right] \\ & \leq \mathbf{E} \left[ \int_0^T |G^{1n}(s, X_s^{0,x}, \bar{Y}_s^{1n}, \bar{Y}_s^{2n}, \bar{Z}_s^{1n}, \bar{Z}_s^{2n}) - G^1(s, X_s^{0,x}, \bar{Y}_s^{1n}, \bar{Y}_s^{2n}, \bar{Z}_s^{1n}, \bar{Z}_s^{2n})| \cdot \right. \\ & \quad \cdot \mathbb{1}_{\{\frac{1}{k} < |\bar{Y}_s^{1n}| + |\bar{Y}_s^{2n}| + |\bar{Z}_s^{1n}| + |\bar{Z}_s^{2n}| < k\}} ds \Big] \\ & \quad + \mathbf{E} \left[ \int_0^T |G^{1n}(s, X_s^{0,x}, \bar{Y}_s^{1n}, \bar{Y}_s^{2n}, \bar{Z}_s^{1n}, \bar{Z}_s^{2n}) - G^1(s, X_s^{0,x}, \bar{Y}_s^{1n}, \bar{Y}_s^{2n}, \bar{Z}_s^{1n}, \bar{Z}_s^{2n})| \cdot \right. \\ & \quad \cdot \mathbb{1}_{\{|\bar{Y}_s^{1n}| + |\bar{Y}_s^{2n}| + |\bar{Z}_s^{1n}| + |\bar{Z}_s^{2n}| \geq k\}} ds \Big] \\ & \quad + \mathbf{E} \left[ \int_0^T |G^{1n}(s, X_s^{0,x}, \bar{Y}_s^{1n}, \bar{Y}_s^{2n}, \bar{Z}_s^{1n}, \bar{Z}_s^{2n}) - G^1(s, X_s^{0,x}, \bar{Y}_s^{1n}, \bar{Y}_s^{2n}, \bar{Z}_s^{1n}, \bar{Z}_s^{2n})| \cdot \right. \\ & \quad \cdot \mathbb{1}_{\{|\bar{Y}_s^{1n}| + |\bar{Y}_s^{2n}| + |\bar{Z}_s^{1n}| + |\bar{Z}_s^{2n}| \leq \frac{1}{k}\}} ds \Big] \\ & \quad + \mathbf{E} \left[ \int_0^T |G^1(s, X_s^{0,x}, \bar{Y}_s^{1n}, \bar{Y}_s^{2n}, \bar{Z}_s^{1n}, \bar{Z}_s^{2n}) - G^1(s, X_s^{0,x}, \bar{Y}_s^1, \bar{Y}_s^2, \bar{Z}_s^1, \bar{Z}_s^2)| ds \right] \\ & := I_1^n + I_2^n + I_3^n + I_4^n, \end{aligned} \tag{4.20}$$

where the sequence  $I_1^n$ ,  $n \geq 1$  converges to 0. On one hand, for  $n \geq 1$ , the point (4.5)-(b) in Step 1 implies that,

$$\begin{aligned} & |G^{1n}(s, X_s^{0,x}, \bar{Y}_s^{1n}, \bar{Y}_s^{2n}, \bar{Z}_s^{1n}, \bar{Z}_s^{2n}) - G^1(s, X_s^{0,x}, \bar{Y}_s^{1n}, \bar{Y}_s^{2n}, \bar{Z}_s^{1n}, \bar{Z}_s^{2n})| \\ & \quad \cdot \mathbb{1}_{\{\frac{1}{k} < |\bar{Y}_s^{1n}| + |\bar{Y}_s^{2n}| + |\bar{Z}_s^{1n}| + |\bar{Z}_s^{2n}| < k\}} \\ & < C_f C_\sigma (1 + |X_s^{0,x}|)k + C_h (1 + |X_s^{0,x}|^\gamma)k. \end{aligned}$$

On the other hand, considering the point (4.5)-(d), we obtain that,

$$\begin{aligned} & |G^{1n}(s, X_s^{0,x}, \bar{Y}_s^{1n}, \bar{Y}_s^{2n}, \bar{Z}_s^{1n}, \bar{Z}_s^{2n}) - G^1(s, X_s^{0,x}, \bar{Y}_s^{1n}, \bar{Y}_s^{2n}, \bar{Z}_s^{1n}, \bar{Z}_s^{2n})| \\ & \quad \cdot \mathbb{1}_{\{\frac{1}{k} < |\bar{Y}_s^{1n}| + |\bar{Y}_s^{2n}| + |\bar{Z}_s^{1n}| + |\bar{Z}_s^{2n}| < k\}} \\ \leq & \sup_{\substack{(y_s^1, y_s^2, z_s^1, z_s^2) \\ \frac{1}{k} < |y_s^1| + |y_s^2| + |z_s^1| + |z_s^2| < k}} |G^{1n}(s, X_s^{0,x}, y_s^1, y_s^2, z_s^1, z_s^2) - G^1(s, X_s^{0,x}, y_s^1, y_s^2, z_s^1, z_s^2)| \rightarrow 0, \end{aligned}$$

as  $n \rightarrow \infty$ . Thanks to Lebesgue's dominated convergence theorem, the sequence  $I_1^n$  of (4.20) converges to 0 in  $\mathcal{H}_T^1(\mathbf{R})$ .

The sequence  $I_2^n$  in (4.20) is bounded by  $\frac{C}{k^{\frac{2(q-1)}{q}}}$  with  $q \in (1, 2)$ . Actually, from point (4.5)-(b) and Markov's inequality, for  $q \in (1, 2)$ , we get,

$$\begin{aligned} I_2^n & \leq C \left\{ \mathbf{E} \left[ \int_0^T (1 + |X_s^{0,x}|)^q |\bar{Z}_s^{1n}|^q + (1 + |X_s^{0,x}|^\gamma)^q |\bar{Y}_s^{1n}|^q ds \right] \right\}^{\frac{1}{q}} \times \\ & \quad \times \left\{ \mathbf{E} \left[ \int_0^T \mathbb{1}_{\{|\bar{Y}_s^{1n}| + |\bar{Y}_s^{2n}| + |\bar{Z}_s^{1n}| + |\bar{Z}_s^{2n}| \geq k\}} ds \right] \right\}^{\frac{q-1}{q}} \\ & \leq C \left\{ \mathbf{E} \left[ \int_0^T |\bar{Z}_s^{1n}|^2 ds \right] + \mathbf{E} \left[ \int_0^T (1 + |X_s^{0,x}|)^{\frac{2q}{2-q}} ds \right] \right. \\ & \quad \left. + \mathbf{E} \left[ \int_0^T |\bar{Y}_s^{1n}|^2 ds \right] + \mathbf{E} \left[ \int_0^T (1 + |X_s^{0,x}|)^{\gamma \cdot \frac{2q}{2-q}} ds \right] \right\}^{\frac{1}{q}} \times \\ & \quad \times \frac{\left\{ \mathbf{E} \left[ \int_0^T |\bar{Y}_s^{1n}|^2 + |\bar{Y}_s^{2n}|^2 + |\bar{Z}_s^{1n}|^2 + |\bar{Z}_s^{2n}|^2 ds \right] \right\}^{\frac{q-1}{q}}}{(k^2)^{\frac{q-1}{q}}} \\ & \leq \frac{C}{k^{\frac{2(q-1)}{q}}}. \end{aligned}$$

The last inequality is a straightforward result of the estimates (3.2)(4.12) and (4.13).

The third sequence  $I_3^n$  in (4.20) is bounded by  $C/k$  with constant  $C$  independent on  $k$ . Indeed, by (4.5)-(b) and (3.6),

$$I_3^n \leq \mathbf{E} \left[ \int_0^T C_f C_\sigma (1 + |X_s^{0,x}|) \frac{1}{k} + C_h (1 + |X_s^{0,x}|^\gamma) \frac{1}{k} ds \right] \leq C/k.$$

The fourth sequence  $I_4^n$ ,  $n \geq 1$  in (4.20) also converges to 0, at least for a subsequence. Actually, since the sequence  $(\bar{Z}^{1n})_{n \geq 1}$  converges to  $\bar{Z}^1$  in  $\mathcal{H}_T^2(\mathbf{R}^m)$ , then there exists a subsequence  $(\bar{Z}^{1n_k})_{k \geq 1}$  such that it converges to  $\bar{Z}^1$ ,  $dt \otimes d\mathbf{P}$ -a.e., and furthermore,

$\sup_{k \geq 1} |\bar{Z}_t^{1n_k}(\omega)| \in \mathcal{H}_T^2(\mathbf{R})$ . On the other hand,  $(\bar{Y}^{1n_k})_{k \geq 1}$  converges to  $\bar{Y}^1 > 0$ ,  $dt \otimes d\mathbf{P}$ -a.e.. Thus, taking the continuity of function  $G^1(t, x, y^1, y^2, z^1, z^2)$  w.r.t  $(y^1, y^2, z^1, z^2)$  into account, we obtain that

$$G^1(t, X_t^{0,x}, \bar{Y}_t^{1n_k}, \bar{Y}_t^{2n_k}, \bar{Z}_t^{1n_k}, \bar{Z}_t^{2n_k}) \rightarrow_{k \rightarrow \infty} G^1(t, X_t^{0,x}, \bar{Y}_t^1, \bar{Y}_t^2, \bar{Z}_t^1, \bar{Z}_t^2) \quad dt \otimes d\mathbf{P} - a.e.$$

In addition, considering that

$$\sup_{k \geq 1} |G^1(t, X_t^{0,x}, \bar{Y}_t^{1n_k}, \bar{Y}_t^{2n_k}, \bar{Z}_t^{1n_k}, \bar{Z}_t^{2n_k})| \in \mathcal{H}_T^q(\mathbf{R}) \text{ for } 1 < q < 2,$$

which follows from (4.15). Finally, by the dominated convergence theorem, one can get that,

$$G^1(t, X_t^{0,x}, \bar{Y}_t^{1n_k}, \bar{Y}_t^{2n_k}, \bar{Z}_t^{1n_k}, \bar{Z}_t^{2n_k}) \rightarrow_{k \rightarrow \infty} G^1(t, X_t^{0,x}, \bar{Y}_t^1, \bar{Y}_t^2, \bar{Z}_t^1, \bar{Z}_t^2) \quad \text{in } \mathcal{H}_T^q(\mathbf{R}),$$

which yields to the convergence of  $I_4^n$  in (4.20) to 0.

It follows that the sequence  $(G^{1n}(t, X_t^{0,x}, \bar{Y}_t^{1n}, \bar{Y}_t^{2n}, \bar{Z}_t^{1n}, \bar{Z}_t^{2n})_{0 \leq t \leq T})_{n \geq 1}$  converges to  $(G^1(t, X_t^{0,x}, \bar{Y}_t^1, \bar{Y}_t^2, \bar{Z}_t^1, \bar{Z}_t^2))_{0 \leq t \leq T}$  in  $\mathcal{L}^1([0, T] \times \Omega, dt \otimes d\mathbf{P})$  and then  $F^1(t, X_t^{0,x}) = G^1(t, X_t^{0,x}, \bar{Y}_t^1, \bar{Y}_t^2, \bar{Z}_t^1, \bar{Z}_t^2)$ ,  $dt \otimes d\mathbf{P}$ -a.e. In the same way, we have,  $F^2(t, X_t^{0,x}) = G^2(t, X_t^{0,x}, \bar{Y}_t^1, \bar{Y}_t^2, \bar{Z}_t^1, \bar{Z}_t^2)$ ,  $dt \otimes d\mathbf{P}$ -a.e. Thus, the processes  $(Y^i, Z^i)$ ,  $i = 1, 2$  are the solutions of the backward equation (4.4).

**Step 6.** The solutions  $(Y_t^i, Z_t^i)$ ,  $i = 1, 2$  for BSDE (4.2) exist.

Obviously observed from (4.11) that  $\bar{Y}_t^1$  is strict positive which enable us to obtain the solution of the original BSDE (4.2) by:

$$\begin{cases} Y_t^1 = \ln \bar{Y}_t^1; \\ Z_t^1 = \frac{\bar{Z}_t^1}{\bar{Y}_t^1}, \quad t \in [0, T]. \end{cases}$$

The same illustrate about the case  $i = 2$  gives the existence of solution  $(Y^2, Z^2)$  for BSDE (4.2). Besides, it follows from the fact  $\bar{Y}_t^i = \varsigma^i(t, X_t^{0,x})$  and for each  $(t, x) \in [0, T] \times \mathbf{R}^m$ ,  $e^{-C(1+|x|^\gamma)} \leq \varsigma^i(t, x) \leq e^{C(1+|x|^\gamma)}$  with  $1 < \gamma < 2$ , that  $Y^i$  also has a representation through a deterministic function  $\varpi^i(t, x) = \ln \varsigma^i(t, x)$  which is of subquadratic growth, i.e.  $|\varpi^i(t, x)| \leq C(1 + |x|^\gamma)$  with  $1 < \gamma < 2$ ,  $i = 1, 2$ . The proof is completed.  $\square$

## References

- [1] D.G. Aronson, *Bounds for the fundamental solution of a parabolic equation*, Bulletin of the American Mathematical society, 73.6, (1967), pp. 890-896.
- [2] P. Barrieu, N. El-Karoui, *Monotone stability of quadratic semimartingales with applications to unbounded general quadratic BSDEs*, The Annals of Probability, 41.3B, (2013), pp. 1831-1863.

- [3] T. Başar, *Nash equilibria of risk-sensitive nonlinear stochastic differential games*, Journal of optimization theory and applications 100.3, (1999), pp. 479-498.
- [4] C. Doléan-Dade, C. Dellacherie, and P. A. Meyer, *Diffusions à coefficients continus, d'après Stroock et Varadhan*, Séminaire de Probabilités (Strasbourg), 4, (1970), pp. 240-282.
- [5] N. El-Karoui and S. Hamadène, *BSDEs and risk-sensitive control, zero-sum and nonzero-sum game problems of stochastic functional differential equations*, Stochastic Processes and their Applications, 107.1, (2003), pp. 145-169.
- [6] N. El-Karoui, S. Peng and M.C. Quenez, *Backward stochastic differential equations in finance*, Mathematical finance, 7.1, (1997), pp. 1-71.
- [7] W.H. Fleming, *Risk sensitive stochastic control and differential games*, Communications in Information & Systems, 6.3, (2006), pp. 161-177.
- [8] W.H. Fleming, W.M. McEneaney, *Risk sensitive optimal control and differential games*, Springer Berlin Heidelberg, (1992).
- [9] W.H. Fleming, W.M. McEneaney, *Risk-sensitive control on an infinite time horizon*, SIAM Journal on Control and Optimization, 33.6 (1995), pp 1881-1915.
- [10] A. Friedman, *Differential games*, Wiley, New York, (1971).
- [11] I.V. Girsanov, *On transforming a certain class of stochastic processes by absolutely continuous substitution of measures*, Theory of Probability and its Applications, 5, (1960), pp. 285-301.
- [12] S. Hamadène, *Backwardforward SDEs and stochastic differential games*, Stochastic processes and their applications, 77.1, (1998), pp. 1-15.
- [13] S. Hamadène, J.-P. Lepeltier and S. Peng, *BSDEs with continuous coefficients and stochastic differential games*, Pitman Research Notes in Mathematics Series, (1997), pp. 115-128.
- [14] U.G. Haussmann, *A stochastic maximum principle for optimal control of diffusions*, John Wiley & Sons, Inc. (1986).
- [15] Matthew R. James, *Asymptotic analysis of nonlinear stochastic risk-sensitive control and differential games*, Mathematics of Control, Signals and Systems, 5.4, (1992), pp. 401-417.
- [16] I. Karatzas and S.E. Shreve, *Brownian Motion and Stochastic Calculus - 2nd ed.*, Springer Verlag, (1991).
- [17] M. Kobylanski, *Backward stochastic differential equations and partial differential equations with quadratic growth*, The Annals of Probability, 28.2, (2000), pp. 558-602.
- [18] E. Pardoux and S. Peng, *Adapted solution of a backward stochastic differential equation*, Systems & Control Letters, 14.1, (1990), pp. 55-61.

- [19] E. Pardoux and S. Peng, *Backward stochastic differential equations and quasilinear parabolic partial differential equations*, Stochastic partial differential equations and their applications, Springer Berlin Heidelberg,(1992), pp. 200-217.
- [20] S. Peng, *Backward stochastic differential equation, nonlinear expectation and their applications*, Proceedings of the International Congress of Mathematicians, (2011), pp. 393-432.
- [21] P. Protter, *Stochastic Integration and Differential Equations, 2nd ed.*, Springer-Verlag, (2004).
- [22] D. Revuz and M. Yor, *Continuous martingales and Brownian motion*, 293, Springer Verlag, (1999).
- [23] H. Tembine, Q. Zhu, T. Başar, *Risk-sensitive mean-field stochastic differential games*, Proc. 18th IFAC World Congress, (2011).